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AN

INTRODUCTORY ALGEBRA

BY

JOHN H. WALSH

ASSOCIATE SUPERINTENDENT OF SCHOOLS
THE CITY OF NEW YORK

D. C. HEATH & CO., PUBLISHERS
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PREFACE

To the pupil who remains at school until the middle of the tenth year, the so-called logical arrangement of the topics of elementary algebra does no particular harm. Since, however, the majority of boys and girls that begin this study drop out before its completion, it is important to secure for them the most profitable use of the time they can devote to it.

Algebra owes its place in the curriculum mainly for two reasons; the chief one being its practical value in the solution of numerical problems in many departments of human activity, and the other being the necessity of an acquaintance with algebra before taking up the study of the higher mathematics.

All of algebra that can be used in after life by the pupil that terminates his mathematical studies with this branch is contained in the equation work in its simpler forms. On this account it is the duty of the school authorities to see that this part of the work receives earliest attention, whether algebra is begun in the high school or in the grammar grades, and that all types of equations should be presented to the pupils' attention before much time is assigned to the drills in the fundamental processes.

The INTRODUCTORY ALGEBRA is intended to cover somewhat less than one half of the work generally prescribed for the course in elementary algebra. This it does, not by taking one half of the topics, but by including the simplest and most important part of each topic. The book consists of two parts; PART ONE, devoted almost entirely to equations and problems, furnishing in brief compass a reasonable familiarity with the more useful forms. In PART TWO the work in equations is extended to examples slightly more difficult, and a systematic treatment of exercises in all of the fundamental operations is supplied, dealing, however, chiefly with binomial expressions.

The book provides a maximum of practice with a minimum of theory, the assumption being that the latter is out of place at this stage of the work. Explanatory matter is supplied to be read and discussed in the class, but it is not intended that any statements should be memorized. These are given merely to enable pupils to employ technical terms correctly. Adequate definitions of the latter should not be expected until much later in the course.

In order that time may be found for the solution of a large number of problems, the conditions are limited to such as are within the experience of the average pupil. When the opposite is the case, the problem should be omitted. It is inadvisable to devote time assigned to algebra to instruction in physics, agriculture, and the like. The power developed by solving problems involving familiar conditions will enable the pupil to solve similar types later when he has become acquainted with the new conditions.

Throughout the book are interspersed numerous sight exercises including many arithmetical ones. The written work includes algebraic solutions of certain classes of arithmetical problems.

The INTRODUCTORY ALGEBRA supplies an adequate treatment of the subject for pupils of elementary schools. Where time does not permit the completion of all the exercises, those indicated as "Supplementary Work" may be omitted, at least by the slower pupils.

In the preparation of the work, the pupil has been kept constantly in mind, and every detail has been considered from his standpoint. The aim has been to give the subject such a treatment as will secure the greatest practical value to the pupil discontinuing his mathematical studies at this point, and at the same time giving other students all the benefits obtainable in the time devoted to the subject. The latter will be enabled to continue their work more intelligently and with greater interest after having mastered the work presented thus far. Many a pupil becomes discouraged at his inability to overcome the numerous difficulties presented in the ordinary treatment of elementary algebra, especially when he does not realize the ultimate purpose of its study or its availability in lightening his arithmetical burden.

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ALGEBRA

PART ONE

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SIMPLE EQUATIONS

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ORAL PROBLEMS

1. After buying 20 sheep, a farmer has 50 sheep. How many had he at first?

2. John spent $30 for a suit of clothes and then had $20. How much money had he before buying the clothes?

3. How many cherries did Mary give away if she gave away all but 20 of the 50 she had picked?

4. What number gives a product of 88 when it is multiplied by 4?

5. Mr. Smith has 90 acres in one third of his farm. How many acres are there in the farm?

6. A strip of carpet $\frac{3}{4}$ yd. wide contains 12 square yards. How long is it?

7. At $\$\frac{1}{4}$ per pound, how many pounds of coffee will cost $10?

SIGHT EXERCISES

Give answers:

1. $20 + ? = 50.$
2. $? - 30 = 20.$
3. $\frac{1}{3}$ of $? = 90.$
4. $? \times 4 = 88.$
5. $50 - ? = 30.$
6. $? \times \frac{3}{4} = 12.$

Give value of x:

7. $x + 20 = 50.$
8. $x - 30 = 20.$
9. $50 - x = 30.$
10. $3x = 40 + 20.$
11. $4x = 90 - 10.$
12. $x + 3x = 80.$
13. $x + 3x = 80 - 20.$
14. $5x - 2x = 80 + 10.$
15. $x + 2x = 60 - 30.$
16. $5x + 2x = 90 - 20.$
17. $3x - x = 40 + 20.$
18. $x + 2x + 3x = 36.$

Each of the foregoing exercises constitutes an **equation.** The parts of the equation connected by the sign of equality are called **members.**

By the **solution** of an equation is meant the process of finding the number represented by the letter.

Thus, the equation $4x = 80$ is solved by dividing both members by 4. This gives the equation $x = 20$, which shows the value of x.

The number whose value is to be determined is called the **unknown number.** It is generally represented by one of the last letters of the alphabet: x, y, z.

Find the value of y in the following equation:

$$4y + 5y - 3y = 90 - 42.$$

METHOD

Given $\quad 4y + 5y - 3y = 90 - 42$

$$6y = 48$$

$$y = 8. \quad \textit{Ans.}$$

Test: When 8 is substituted for y, first member becomes

$$32 + 40 - 24 = 48.$$

Second member, $\quad 90 - 42 = 48.$

Since this value of y makes both members identical, it is correct.

WRITTEN EXERCISES

Find the values of x, y, z, etc.:

1. $x + 2x + 3x = 114.$
2. $5y + 3y = 97 - 49.$
3. $6z - 2z = 119 - 75.$
4. $2x + 4x - 3x = 81.$
5. $7w = 97 + 45 - 23.$
6. $v + 3v - 2v = 66.$
7. $2x + 3x - x = 76.$
8. $4y + 3y - 2y = 90.$
9. $6z - z = 84 - 29.$
10. $9x - x - 2x = 258.$
11. $14v + 10v + 6v = 90.$
12. $27w - 3w = 86 + 10.$
13. $3\frac{1}{2}x + 5\frac{1}{2}x = 117 - 9.$
14. $5y - 3y + 4y = 96.$
15. $17x - x - 5x = 121.$
16. $24z - 3z = 206 + 4.$
17. $3v + 4v + 5v = 108.$
18. $27x + 17x - 30x = 28.$

WRITTEN PROBLEMS

1. The difference between two numbers is 77, and one number is 12 times the other. Find the numbers.

SOLUTION

Let $x =$ smaller number.
Then $12x =$ larger number,
and $\underline{12x - x = \text{difference.}}$
Therefore $12x - x = 77,$
or $11x = 77.$
$x = 7$, the smaller number.
$12x = 84$, the larger number.

Test: $84 - 7 \doteq 77.$ *Ans.* 7 and 84.

Results should always be tested to ascertain if the answers obtained satisfy the conditions of the problem. Since the larger number, 84, is obtained by multiplying the smaller number by 12, one of the conditions is satisfied. As the test shows that the second condition is also satisfied, the answers are correct.

2. Find two numbers whose sum is 91, one of the numbers being 12 times the other.

3. A chair and a desk cost $27, the cost of the latter being 8 times the cost of the former. What was the cost of each?

Solution

Let $x = \text{cost of chair (in dollars)}.$
Then $8x = \text{cost of desk (in dollars)},$
and $\underline{x + 8x = 9x = \text{cost of both}}.$
Therefore $9x = 27.$
$x = 3,$ cost of chair (in dollars).
$8x = 24,$ cost of desk (in dollars).

Test: $\$3 + \$24 = \$27.$ *Ans.* $\$3, \$24.$

Note. Only *abstract* numbers are employed in equations. When the answer is *concrete*, the necessary denomination is supplied in writing the final result.

4. Divide $\$450$ between two persons in such a way that one shall receive 4 times as much as the other.

5. The sum of two numbers is 81, the smaller being one half the larger. Find the numbers.

Suggestion. Let x = smaller; then, $2x$ = larger.

6. A horse and a wagon together cost $\$675$, the cost of the former being one half that of the latter. What was the cost of each?

7. Three farmers have 360 sheep. The second has 3 times as many as the first, and the third has twice as many as the other two together. How many sheep has each?

Suggestion. When x = number owned by the first, then $3x$ = number owned by the second; and $8x$ = number owned by the third; the number owned by all three being $x + 3x + 8x$, or $12x$.

8. A, B, and C have together 720 acres of land. B has twice as much land as A, and C has 3 times as much as A. How many acres has each?

9. Edward has twice as many marbles as John, and James has three times as many as John and Edward together. Find the number of marbles belonging to each, the total number being 72.

Suggestion. x, $2x$, $9x$.

10. The sum of the ages of a father and his two sons is 63 years, the father's age being 16 times that of the younger son and 4 times that of the older son. How old is each?

Suggestion. When x represents age of younger son in years, $16x$ represents the father's age, and $16x \div 4$, or $4x$, represents age of older son.

11. A mother's age is 8 times her daughter's, and the difference of their ages is $24\frac{1}{2}$ years. How old is each?

12. Divide $648 among A, B, and C, giving B twice as much as A, and C twice as much as A and B together.

13. The perimeter of a rectangle measures 136 yards, the length being 3 times the width. Find the dimensions.

Suggestion. x, $3x$, x, $3x$.

14. A, B, and C receive $720 for doing a piece of work. What does each receive if B is paid twice as much as C, and A is paid $1\frac{1}{2}$ times as much as B?

15. The sum of three numbers is 120, the first being twice the second, and the third being 3 times the sum of the other two. What are the numbers?

16. A woman has $119 in 2-dollar and 5-dollar bills, the same number of each denomination. How many bills has she?

Solution

Let x = number of each.
Then $2x$ = value of $2 bills,
and $5x$ = value of $5 bills.
Therefore $7x = 119$, etc.
Find the whole number of bills; $2x$. *Test.*

17. A boy buys an equal number of 1¢, 2¢, and 5¢ postage stamps at a cost of $1.12. How many of each variety does he buy?

Suggestion. $x + 2x + 5x = 112$.

Note. $1.12 is changed to 112 cents, the price of the stamps being expressed in cents.

18. Two trains starting at the same time from points 284 miles apart, travel towards each other, one at the rate of 38 miles per hour and the other at the rate of 33 miles per hour. In how many hours will they meet?

SUGGESTION. If x represents the number of hours traveled by each to the meeting place, $38x + 33x$ will represent the number of miles traveled by both, or 284.

19. The yields of two fields of the same size amounted to 1400 bushels, one field producing 16 bushels of wheat to the acre, and the other 54 bushels of corn to the acre. Find the size of each field.

20. The difference between two numbers is 169, and the less is one fourteenth of the greater. Find the numbers.

Let x = smaller.

21. The sum of $375 is divided between M and N, the difference between their respective shares being 3 times N's share. What does each receive?

22. A drover expends $1030 in the purchase of sheep at $3 each, pigs at $5 each, and cows at $30 each, buying twice as many pigs as sheep, and as many cows as pigs and sheep together. How many does he buy of each?

Respective number of each kind: x, $2x$, $3x$.
Respective cost of each kind: $3x$, $10x$, $90x$.

23. A man mixed 282 bushels of corn and oats, using twice as much of the former as of the latter. How many bushels of corn does he use?

24. Two trains start at the same time from cities 240 miles apart, and travel towards each other, one going twice as fast as the other. How far must the former travel before meeting the other?

25. A has three times as many stamps as B, and B has six times as many as C. They have, in all, 1000 stamps. How many has each?

Clearing of Fractions

ORAL EXERCISES

1. One third of a number is 21. Find the number.

2. One third of a number is 21. Find two thirds of the number.

3. If two thirds of a number is 18, what is the number?

4. Sixty acres is three fourths of how many acres?

5. If 60 is 3 fourths of a number, what is 1 fourth of the same number?

6. The sum of one half of a number and one fourth of the same number is 24. What is the number?

7. A number diminished by one third of itself is 18. Find the number.

8. After giving away one fourth of his marbles, John has 24 remaining. How many had he at first?

SIGHT EXERCISES

Give the value of x, y, z, etc.:

Note. $\frac{x}{7}$ means the same as $\frac{1}{7}x$:

1. $\frac{x}{7}=2.$
2. $\frac{y}{3}=5.$
3. $\frac{z}{4}=8.$
4. $\frac{v}{5}=9.$
5. $\frac{w}{6}=7.$
6. $\frac{6x}{7}=12.$
7. $\frac{2y}{3}=10.$
8. $\frac{3z}{4}=24.$
9. $\frac{4v}{5}=36.$
10. $\frac{5w}{6}=35.$
11. $\frac{6x}{7}=18.$
12. $\frac{2y}{3}=24.$
13. $\frac{3z}{4}=36.$
14. $\frac{4v}{5}=20.$
15. $\frac{5w}{6}=30.$
16. $x-\frac{x}{2}=12.$
17. $x-\frac{x}{3}=12.$
18. $x-\frac{x}{4}=12.$
19. $x-\frac{x}{5}=12.$
20. $x-\frac{x}{7}=12.$

WRITTEN EXERCISES

Find the value of x, y, z, etc.:

1. $\frac{x}{7}=43.$
2. $\frac{y}{3}=57.$
3. $\frac{z}{4}=39.$
4. $\frac{v}{5}=72.$
5. $\frac{w}{6}=28.$
6. $\frac{6x}{7}=258.$
7. $\frac{2y}{3}=114.$
8. $\frac{3z}{4}=231.$
9. $\frac{4v}{5}=260.$
10. $\frac{5w}{6}=135.$
11. $x-\frac{x}{7}=84.$
12. $y-\frac{y}{3}=42.$
13. $z-\frac{z}{4}=36.$
14. $v-\frac{v}{5}=60.$
15. $w-\frac{w}{6}=90.$

To find the value of x, when $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94.$

SOLUTION

Given $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94.$

Multiplying each term in both members by 60 (L. C. M. of 3, 4, 5),

$$20x+15x+12x=5640.$$

Combining, $47x=5640.$

$$x=120. \quad \textit{Ans.}$$

Test: Substituting 120 for x, first member becomes

$$\frac{120}{3}+\frac{120}{4}+\frac{120}{5}=40+30+24=94.$$

16. $\frac{2x}{3}-\frac{x}{4}=60+15.$
17. $x-\frac{x}{3}-\frac{x}{4}=75.$
18. $x+\frac{x}{2}+\frac{x}{3}=44.$
19. $5x+\frac{x}{4}-x=68.$
20. $x+\frac{x}{3}-\frac{x}{2}=25.$
21. $x+\frac{4x}{11}+\frac{6x}{11}=84.$
22. $2x-\frac{x}{8}-\frac{x}{9}=127.$
23. $\frac{2x}{3}-\frac{x}{2}+\frac{x}{4}=75.$
24. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=39.$
25. $\frac{x}{2}+\frac{2x}{3}+\frac{3x}{4}=69.$

WRITTEN PROBLEMS

1. A number increased by one half and one third of itself equals 88. What is the number?

$$x + \frac{x}{2} + \frac{x}{3}.$$

2. The sum of two numbers is 32, and one is $\frac{3}{5}$ of the other. Find the numbers.

$$x + \frac{3x}{5}.$$

3. A and B own together 375 acres of land, A having $2\frac{3}{4}$ times as much land as B. How many acres has each?

$$x + \frac{11x}{4}.$$

4. The perimeter of a rectangular field is 300 rods, the length of one side being $\frac{2}{3}$ that of the other. Find the dimensions.

5. Each year a man's salary was increased $\frac{1}{10}$ over the sum he received the preceding year, his total salary for the three years being $ 3310. What was his salary each year?

Let $x =$ 1st year's salary.

Then $\frac{11x}{10} =$ 2d year's salary.

And $\frac{121x}{100} =$ 3d year's salary.

6. The cost of a quantity of wheat together with a commission of $\frac{1}{50}$ of the cost of the wheat amounted to $ 306. What was the cost of the wheat?

7. A man sold a horse for $ 350, thereby gaining $\frac{3}{25}$ of the cost of the horse. How much was his gain?

8. After selling $\frac{1}{4}$ of his apples and $\frac{2}{3}$ of the remainder a boy has 36 apples. How many had he at first?

$\frac{x}{4}$ represents the number first sold, the remainder then being $\frac{3x}{4}$. Two thirds of $\frac{3x}{4}$ is $\frac{x}{2}$, which represents the number next sold.

$$x - \frac{x}{4} - \frac{x}{2} = 36.$$

9. A merchant deducted $\frac{1}{4}$ of the marked price of a piano and made a second deduction of $\frac{1}{10}$ of the remainder, receiving $405 for it. What was the marked price?

10. One third of a number, added to one quarter of the same number, equals 84. Find the number.

11. Four times a boy's money added to one fourth of it is 68 cents. How much money has he?

12. After grinding $\frac{1}{3}$ of his crop of wheat, selling $\frac{1}{4}$ of it, and retaining $\frac{1}{5}$ of it for seed, Mr. A has 104 bushels to sell. How many bushels were there in the crop?

$$x-\frac{x}{3}-\frac{x}{4}-\frac{x}{5}=104.$$

13. Find a number such that the difference between $\frac{1}{8}$ of it and $\frac{1}{9}$ of it is 22.

14. A farmer has one half of his land in corn, one fourth of it in rye, and one eighth in wheat, 280 acres in these three crops. How many acres are there in the farm?

15. A man loaned x dollars at 5% for 2 years, and at the end of the time received $220 principal and interest. How much did he lend?

$$x+\frac{x}{10}.$$

16. Three fourths of five sixths of a number equals 65. Find the number.

17. Two men have 143 sheep, one having $4\frac{1}{2}$ times as many as the other. How many sheep has each?

18. The difference between $37\frac{1}{2}$% of a man's money and 15% of it is $81. How much money has he?

$$\frac{3x}{8}-\frac{3x}{20}=81.$$

19. Divide $280 between two persons so that one will have $\frac{3}{4}$ as much as the other.

20. A's share of 195 bushels of corn is $\frac{3}{4}$ of B's share, and B's share is $\frac{2}{3}$ of C's share. Find the share of each.

Transposition

ORAL EXERCISES

1. If $x + 14 = 24$, what is the value of x?

2. Find the value of $3x$ when $14 + 3x = 44$. What is the value of x?

3. What is the value of $\frac{1}{2}x$ when $x = 10 + \frac{1}{2}x$? What is the value of x?

4. Find the value of x when $x = 9 + \frac{1}{4}x$.

5. When $3x - 17 = 10$, what is the value of $3x$? Of x?

6. Given the equation $4x + 17 = 37$, to find the value of x.

By taking 17 from each member,

$$4x = 37 - 17,$$

or

$$4x = 20,$$

etc.

7. Find the value of x when $4x - 17 = 31$.

If $4x$ diminished by 17 equals 31, $4x$ equals 31 and 17; that is,

$$4x = 31 + 17,$$

or

$$4x = 48,$$

etc.

8. $3x + 15 = 2x + 20$.

Collecting the unknown numbers in one member of the equation and the known numbers in the other,

$$3x - 2x = 20 - 15$$

etc.

In Example 6, the equality of both members of the equation is retained by subtracting 17 from each.

In Example 7, the equality of the members is retained by adding 17 to each.

In Example 8, 15 and $2x$ are taken from each member.

The transfer of a quantity from one member of an equation to the other is called **transposition**.

When a quantity is transposed, its sign is changed.

SIGHT EXERCISES

Give value of unknown quantity:

1. $4x = 3x + 30.$
2. $7x = 6x + 18.$
3. $4x = 2x + 12.$
4. $7x = 5x + 14.$
5. $7x - 10 = 11.$
6. $4x - 10 = 14.$
7. $6x + 9 = 39.$
8. $5x = x + 28.$
9. $6x = 3x + 21.$
10. $9x = 40 - x.$
11. $4x + 20 = 3x + 30.$
12. $4x + 17 = 3x + 25.$
13. $4x - 12 = 2x + 12.$
14. $5x - 10 = 3x + 12.$
15. $6x - 13 = 11 + 2x.$
16. $5x + 13 = 2x + 37.$
17. $6x - 15 = 2x + 21.$
18. $4x - 21 = 3x - 17.$
19. $15 + 3x = 21 + 2x.$
20. $15 - 4x = 21 - 5x.$

WRITTEN EXERCISES

Find value of x. Test each answer:

1. $18x + 6 = 12x + 24.$
2. $18x + 34 = 16x + 48.$
3. $36x - 18 = 9x + 36.$
4. $15x - 51 = 9x + 123.$
5. $42x - 21 = 63 + 14x.$
6. $49x - 35 = 35x + 21.$
7. $75x - 100 = 10x + 30.$
8. $10x - 48 = 48 - 2x.$
9. $128 + 16x = 164 + 12x.$
10. $8x - x = 3x + 16.$

Note. Transpose the unknown numbers to the first member and the known numbers to the second. Before clearing of fractions, combine the known numbers.

11. $\frac{5x}{2} - 7 = \frac{2x}{3} + 4.$
12. $6x - 16 = \frac{3x}{4} + 5.$
13. $6x - 12 = \frac{3x}{4} + 9.$
14. $\frac{3x}{4} - 12 = \frac{x}{7} + 39.$
15. $\frac{5x}{9} + 5 = \frac{x}{3} + 7.$
16. $\frac{3x}{2} + 3 = 22 - 8x.$
17. $4x - \frac{7}{4} = 3 - \frac{3x}{4}.$
18. $\frac{2x}{3} + x - 20 = 0.$
19. $\frac{6x}{5} + 3 = \frac{2x}{3} + 11.$
20. $x - 52 = \frac{x}{3} + \frac{x}{4} + \frac{x}{5}.$

WRITTEN PROBLEMS

1. Divide $60 among three persons, giving the second $5 more than the first, and the third $10 more than the first.

SOLUTION

Let $x =$ share of the first.
Then $x + 5 =$ share of the second,
and $x + 10 =$ share of the third.
Therefore $3x + 15 = 60.$
Transposing and combining $3x = 60 - 15 = 45.$
Dividing by 3 $x = 15$, etc. etc. *Test.*

2. Find a number whose product by 5 exceeds 24 as much as 24 exceeds the number itself.

$$5x - 24 = 24 - x.$$

3. The sum of the ages of three persons is 93 years. The third is 6 years older than the second, and the latter is 9 years older than the first. Find the age of each.

Let $x =$ age of first.

4. Three bins contain together 4770 bushels of grain. The first contains 730 bushels more than the second, and the second contains 520 bushels more than the third. How many bushels are there in each bin?

5. Divide 60 marbles between two boys, giving one 12 marbles fewer than twice the share of the other.

6. In an election in which 5760 votes were cast for two candidates, the successful one received a majority of 284 votes. How many votes did he receive?

7. Three times a number diminished by 74 is equal to twice the number increased by 56. Find the number.

8. After selling one fourth of his potatoes, a farmer has 270 bushels more than he sold. How many bushels did he sell?

$$x - \frac{x}{4} = \frac{x}{4} + 270.$$

9. A and B raised together 1920 bushels of wheat. The difference between their shares is 4 times A's share. What is B's share?

$x =$ A's share; $5x =$ B's share. Why?

10. The sum of two numbers is 159. The larger number is 6 less than 4 times the smaller. Find the numbers.

11. A man traveled 285 miles in three days, going 10 miles more on the second day than on the first, and 10 miles more on the third day than on the second. How many miles did he travel each day?

12. The successful candidate in an election received 215 votes more than the total of both of his competitors, one of whom received 100 votes more than the other. What did each receive, the total vote for the three being 1927?

13. After buying 40 sheep, a farmer had $30 left. In order to buy 60 sheep he would have needed $30 more than he had. What was the cost of a sheep?

$$40x + 30 = 60x - 30.$$

14. A boy sold 255 newspapers in 6 days, each day after the first selling 5 more than on the preceding day. How many did he sell on the sixth day?

15. If a man is 24 years old when his son is born, in how many years will he be three times as old as his son?

16. A and B own together 416 acres of land, A's portion containing 11 acres more than $\frac{1}{8}$ of B's portion. How many acres has each?

17. A certain number consists of two digits, the figure in the tens' place being double the other. If 27 be subtracted from the number, the digits will be inverted. What is the number?

Let $x =$ units' digit.
Then $2x =$ tens' digit,
$20x + x =$ number,
$10x + 2x =$ number inverted.

Symbols and Operations

ALGEBRAIC SYMBOLS

Numbers in arithmetic are composed of figures each of which has a definite value; algebraic numbers contain letters, or letters and signs. Algebraic numbers are called **quantities** or **expressions.**

In its simplest form, an algebraic number is represented by one or more letters or by letters and arithmetical numbers not connected by a + or a − sign; as, a, $3\,b$, $4\,x$, $5\,y$, etc. Each of these is called a **simple quantity**, or a **term.**

Each of the two expressions, $a + 3\,b$ and $4\,x - 5\,y^2$, consists of two terms.

The expression, $a + 3\,b - 4\,x$, contains three terms.

The expression, $a - 3\,b + 4\,x + 5\,y^2$, contains four terms.

In the foregoing expressions, 3, 4, and 5 are called **coefficients.**

POSITIVE AND NEGATIVE NUMBERS

Another difference between algebra and arithmetic is the employment by the former of negative numbers. Thus 7, or $+7$, is a **positive** number, the only kind employed in arithmetical calculations; while -7 is **negative.** The hourly rate of a vessel going due north propelled by sail power at the rate of 3 miles per hour, might be indicated as $+3$ miles. If the wind should cease and the vessel should meet a current running due south at the rate of 4 miles per hour, its hourly rate of progress in a northerly direction could be expressed as -4 miles. In the same way, a temperature of 4 degrees **below** zero can be written as $-4°$, a temperature of 60 degrees **above** zero as $+60°$, or merely 60°, the absence of a prefixed sign indicating that the algebraic term is positive.

To indicate this use of the signs, $+x$, or x, may be read *positive x*; $-y$ may be read *negative y*. They may also be read *plus x*, *minus y*.

Algebraic Operations

ALGEBRAIC ADDITION

Addition, in algebra, consists in uniting two or more quantities into the simplest form.

The algebraic sum of the similar terms $4a$ and $5a$ is $9a$; of $-4a$ and $-5a$ is $-9a$; of $-4a$ and $5a$ is a; of $4a$ and $-5a$ is $-a$. The algebraic sum of the dissimilar terms $4a$ and $5b$ is $4a+5b$; of $4a$ and $-5b$ is $4a-5b$; of $-4a$ and $5b$ is $5b-4a$; of $-4a$ and $-5b$ is $-4a-5b$.

The sum of two or more ***similar*** *terms having the same sign is obtained by writing the algebraic sum of their arithmetical coefficients before the common letter and prefixing the common sign. When the signs are unlike, the difference between the sum of the positive and that of the negative coefficients is written before the letter, and the sign of the greater sum is prefixed.*

The sum of two ***dissimilar*** *terms is indicated by writing one term after the other, each term being preceded by its sign.*

The following examples illustrate the rule:

	1.	2.	3.	4.	5.
	$3x$	$-5m$	$-6a$	$3y$	$a+2b$
	$7x$	$-8m$	a	$-10y$	$3a-b$
	$9x$	$-m$	$10a$	$-5y$	$-4a+b$
Sum:	$19x$	$-14m$	$5a$	$-12y$	$2b$

NOTE. The absence of a coefficient indicates that the coefficient is 1; the absence of a sign before a monomial or before the first term of a polynomial indicates that the term is positive.

6. Add $7a-3c$, $6c-3a$, $-8a-7c$, and $a+15c$.

PROCESS

$$\begin{array}{r}
7a-3c\\
-3a+6c\\
-8a-7c\\
a+15c\\
\hline
-3a+11c \quad \textit{Ans.}
\end{array}$$

As a matter of convenience, similar terms may be arranged in columns, negative first terms having the prefixed $-$ sign.

Commencing with the left-hand column, the separate columns are added in accordance with the rule for the addition of similar terms.

The answer may also be written $11c-3a$.

SIGHT EXERCISES

Add the following:

1.	2.	3.	4.	5.	6.
$4a$	$3x$	$-2a$	$-3b$	$4a$	$2y$
$7a$	$2x$	$-5a$	$-7b$	$-3a$	$7y$
$2a$	x	$-9a$	$-8b$	$2a$	$-y$

Give the sum of each of the following:

7.	8.	9.	10.
$x-4$	$3x+4y$	$3a+4c$	$3a+4$
$-6x+9$	$2x-8y$	$2a-c$	$2a-1$
$7x-8$	$-4x-4y$	$-a+2c$	$-a+2$

WRITTEN EXERCISES

Add the following:

1. $4+a$, $6-a$, $3+6a$, $15-5a$, $3+a$, and $2-2a$.
2. $14a-6y$, $7a+y$, $5a-7y$, $9a-11y$, and $a-y$.
3. $16x-16$, $3x-5$, $x-37$, $12x-5$, and $16-3x$.
4. $6m$, $-4n$, $3m-5n$, $4n-2m$, $7n-m$, and $4n$.
5. $5a-b$, $3b+3a$, $4a-5b$, $-5b-a$, $7b$, and $3a$.

Algebraic Subtraction

PRELIMINARY EXERCISES

1. What number must be added to $5a$ to give $8a$ as the sum?

2. What number must be added to $-5a$ to give $-8a$ as the sum?

3. What number must be added to $5a$ to give $-8a$ as the sum?

4. What number must be added to $-5a$ to give $8a$ as the sum?

Give missing numbers:

	5.	6.	7.	8.
Subtrahend	$9b$	$-9b$	$9b$	$-9b$
Difference	$+?$	$+?$	$+?$	$+?$
Minuend	$11b$	$-11b$	$-11b$	$11b$

SIGHT EXERCISES

Give differences:

	1.	2.	3.	4.
Minuend	$8c$	$-8c$	$8c$	$-8c$
Subtrahend	$2c$	$2c$	$-2c$	$-2c$
Difference	?	?	?	?

	5.	6.	7.	8.	9.
From	$3m$	$-6b$	$4a$	$-17x$	$4x-5y$
Subtract	$5m$	$-2b$	$-4a$	$3x$	$-6x-2y$

	10.	11.	12.	13.
From	$3m-6b$	$4a+y$	$-17x+z$	$x+4$
Subtract	$5m-2b$	$-4a-y$	$3x+z$	$-3x-4$

14. (*a*) From 20 take $(19-6)$. (*b*) From $20+6$ take 19. (*c*) What is the value of $20-19+6$?

To subtract one quantity from another combine, as in addition, the terms of the minuend with the terms of the subtrahend after changing the sign of each term of the latter.

WRITTEN EXERCISES

1. From $5a-7$ take $15-3a$.

PROCESS

$$\begin{array}{rr} 5a & -\;7 \\ -3a & +\;15 \\ + & - \\ \hline 8a & -\;22 \end{array} \quad \textit{Ans.}$$

For convenience, the subtrahend is written under the minuend, similar terms being placed in the same column. Under each term of the former is written its changed sign. Using the new signs instead of the original ones, the terms are combined as in addition.

2. $3a-9-(6a-18)$.

$$\begin{array}{rr} 3a & -\;9 \\ 6a & -\;18 \\ - & + \\ \hline -3a & +\;9 \end{array}$$

Ans.

Since the quantity contained in the parenthesis is preceded by a minus sign, it may be written as a subtrahend.

The answer may be written $9-3a$.

3. $4a-6-(2a+7)$.
4. $6b+2-(b-3)$.
5. $m-5-(6-4m)$.
6. $(x+6y)-(6x-y)$.
7. $4x-14-(3x-16)$.
8. $27n-8-(-24)$.
9. $3a+7b-(6a-b)$.
10 $4m-3n-(2m+5n)$.

Addition and Subtraction

An algebraic expression is **simplified** *by performing, as far as possible, all indicated operations.*

WRITTEN EXERCISES

Simplify the following expressions:

1. $2a + 3 - (5a - 4) + (a + 5) - 2a - 6.$

Omitting the parentheses, the given expression is written:

$$2a + 3 - 5a + 4 + a + 5 - 2a - 6.$$

In writing this, the signs of the subtrahend in the first parenthesis are changed, including that of $5a$, the first term.

Combining similar terms, the expression becomes:

$$-4a + 6 \quad \textit{Ans.}, \text{ or } 6 - 4a \quad \textit{Ans.}$$

To test the accuracy of the result, the value of a may be taken as 1. When $a = 1$, the original expression becomes

$$2 + 3 - (5 - 4) + (1 + 5) - 2 - 6 = 2 + 3 - 1 + 6 - 2 - 6 = 2.$$

The answer becomes $-4 + 6 = 2$.

NOTE. A parenthesis preceded by a + sign may be removed without affecting the value of the expression.

2. $4a - 9 + (2a - 7) + 8 - a - (3a - 16).$
3. $3x + 2 - (x - 3) - 6 + 2x - (x - 5).$
4. $m - 5 - (2m + 6) - (10 - m) + 7 + m.$
5. $x + y - (2x - y) + 3x + 5y - (7y - x).$

Multiplication and Division

An algebraic expression seldom contains the arithmetical signs for multiplication and division. The product of a by b is written ab, 4 times x is written $4x$. The product of 6 times the quantity $(2x - 7)$ is indicated by placing 6 before or after a parenthesis containing it, thus: $6(2x - 7)$ or $(2x - 7)6$. The product is $12x - 42$, obtained by multiplying each term of the quantity by 6.

SIGHT EXERCISES

Simplify the following expressions:

1. $6(3x-9)$.
2. $7(8-3x)$.
3. $8(2m+7)$.
4. $9(a+2b)$.
5. $3(4-n)$.
6. $a(b+4)$.
7. $4(a+b)$.
8. $x(2-y)$.
9. $3(x+y)$.
10. $2(ab+3)$.
11. $(m+6n)12$.
12. $(a+9b)8$.
13. $(3x+2y)9$.
14. $(4a-5)10$.
15. $(2-5z)12$.

When it is desired to use a sign to indicate multiplication, the regular sign (×) is employed, or a dot (·) placed above the line.

$$2+3\times ab\equiv 2+3ab, \qquad 2\cdot 3+ab\equiv 6+ab.$$

To emphasize the fact that the second member in each of these expressions is derived directly from the first, the sign of equality may be replaced by the sign of identity ($\equiv$). They may be read thus:

"$2+3\times ab$ is the same as $2+3ab$;
or $2\cdot 3+ab$ is identical with $6+ab$."

The use of the sign of equality is by no means incorrect.

Division is generally indicated by writing the divisor as the denominator of a fraction of which the dividend is the numerator.

SIGHT EXERCISES

Simplify:

1. $\frac{18x-54}{6}$.
2. $\frac{18x-54}{9}$.
3. $\frac{24m+56}{8}$.
4. $\frac{18a+27b}{9}$.
5. $\frac{16-4n}{4}$.
6. $\frac{ab+6a}{a}$.
7. $\frac{10a+12b}{2}$.
8. $\frac{2x-xy}{x}$.
9. $\frac{6x+9y}{3}$.
10. $\frac{2ab+8}{2}$.
11. $\frac{12m+36n}{6}$.
12. $\frac{8a+36b}{4}$.
13. $\frac{40a-50}{10}$.
14. $\frac{27x-36y}{9}$.
15. $\frac{24-48b}{12}$.

WRITTEN EXERCISES

Simplify the following:

1. $6(a-2)-3(2a-6)$.

Process

$$\begin{aligned}6(a-2)-3(2a-6) &\equiv 6a-12-(6a-18),\\ &\equiv 6a-12-6a+18,\\ &\equiv 6. \quad \textit{Ans.}\end{aligned}$$

In simplifying this expression, the terms within the parentheses are first multiplied by the number prefixed. The second product, $6a-18$, is first left within the parenthesis with the minus sign prefixed, after which the parenthesis is removed and the signs of the inclosed terms are changed. The similar terms are then combined.

2. $\frac{18x-54}{6}-\frac{18x-54}{9}+\frac{27}{3}$.

Process

$$\begin{aligned}\frac{18x-54}{6}-\frac{18x-54}{9}+\frac{27}{3} &\equiv 3x-9-(2x-6)+9,\\ &\equiv 3x-9-2x+6+9,\\ &\equiv x+6. \quad \textit{Ans.}\end{aligned}$$

The second quotient, $2x-6$, is placed in a parenthesis to show that the compound term $(2x-6)$ is to be subtracted.

3. $24-(x+5)-28+2(12-2x)$.

4. $15-2(x+3)$.

5. $44-4(8+x)$.

6. $10(x+4)-30$.

7. $30-10(x+4)$.

8. $8(x+40)-6(x-40)$.

9. $8(x-40)-6(x+40)$.

10. $10(3x-4)-4(x+3)$.

11. $30x-2(3x-4)$.

12. $24-2(x+5)-28-2(12-2x)$.

13. $44-4(x+8)-8(5-x)+6(x-3)$.

14. $10(x+4)-30-10(3x-4)-4(x-3)$.

Equations containing Parentheses

WRITTEN EXERCISES

Find the value of x:

1. $24 - (x + 5) = 28 - 2(12 - 3x)$.

METHOD

Given $24 - (x + 5) = 28 - 2(12 - 2x)$.

(*a*) Performing indicated multiplication,

$$24 - (x + 5) = 28 - (24 - 4x).$$

(*b*) Removing parentheses,

$$24 - x - 5 = 28 - 24 + 4x.$$

(*c*) Transposing, $-x - 4x = 28 - 24 - 24 + 5$.

(*d*) Combining, $-5x = -15$.

(*e*) Changing signs, $5x = 15$.

(*f*) Dividing, $x = 3$. *Ans.*

Test: Substituting 3 for x,
first member becomes

$$24 - (3 + 5) \equiv 24 - 8 \equiv 16;$$

second member becomes

$$28 - 2(12 - 6) \equiv 28 - 2 \cdot 6 \equiv 28 - 12 \equiv 16.$$

As the pupils grow proficient they may dispense with step (*a*) by removing the parentheses and multiplying at the same time, and also with (*e*). In the earlier examples it is better to take too many steps than too few.

2. $5(3x - 4) = 2(x + 3)$.
3. $44 + x - 4(8 + x) = 3$.
4. $13(x - 3) = 6(x + 4)$.
5. $8(x - 1) - 2(3x - 2) = 0$.
6. $30(x + 14) = 22(x + 30)$.
7. $10(3x - 4) - 4(x + 3) = 0$.
8. $8(x + 40) = 6(4x + 40)$.
9. $3x + 2(87 - x) = 216$.
10. $8x - 2(x + 10) = 100$.
11. $4(26 - 2x) + 6(x + 1) = 88$.
12. $32 - 9(x - 7) = 4(x + 1)$.
13. $30 - (x + 3) = 18 + 2(9 - 2x)$.
14. $4(x + 40) - 3(4x + 40) = 0$.
15. $30x - 2(3x - 4) = 28x - 12$.

16. $2x-\frac{4x-14}{3}=12-\frac{7x+1}{9}.$

Method

Given $$2x-\frac{4x-14}{3}=12-\frac{7x+1}{9}.$$

(*a*) Multiplying both members by 9,

$$18x-3(4x-14)=108-(7x+1).$$

(*b*) Performing indicated multiplication,

$$18-(12x-42)=108-(7x+1).$$

(*c*) Removing parentheses,

$$18x-12x+42=108-7x-1.$$

(*d*) Transposing, $18x-12x+7x=108-1-42.$

(*e*) Combining, $13x=65.$

(*f*) Dividing, $x=5.$ *Ans.*

Test: Substituting 5 for x,
first member becomes

$$10-\frac{20-14}{3}\equiv 10-\frac{6}{3}\equiv 10-2\equiv 8;$$

second member becomes

$$12-\frac{35+1}{9}\equiv 12-\frac{36}{9}\equiv 12-4\equiv 8.$$

If it be desired to make the fractional terms positive, they may be transposed, the given equation becoming

$$2x+\frac{7x+1}{9}=12+\frac{4x-14}{3}.$$

Multiplying both members by 9:

$$18x+7x+1=108+12x-42.$$

17. $\frac{5x+70}{2}=\frac{11x}{3}.$

18. $\frac{3x-4}{2}=\frac{x+3}{5}.$

19. $x-1=\frac{3x-2}{4}.$

20. $\frac{10x-1}{7}=\frac{5x+17}{6}.$

21. $\frac{3x+1}{4}=\frac{4x-2}{5}.$

22. $\frac{26-2x}{3}=\frac{11x+1}{6}-\frac{x-1}{2}.$

23. $6-\frac{7x+1}{18}=x-\frac{2x-7}{3}.$

24. $x-\frac{5x-5}{9}=\frac{3x+15}{8}.$

25. $\frac{4x+7}{9}=\frac{5x-9}{6}+\frac{1}{3}$.

26. $2x-\frac{10x+3}{11}=\frac{4x+3}{5}$.

27. $\frac{4x-5}{17}=\frac{5x-1}{23}$.

Method

Given $\frac{4x-5}{17}=\frac{5x-1}{23}$

(*a*) Multiplying both members by 17×23,

$$\frac{(4x-5)17\times 23}{17}=\frac{(5x-1)17\times 23}{23}.$$

(*b*) Canceling, $(4x-5)23=(5x-1)17$.

(*c*) Performing indicated multiplications,

$$92x-115=85x-17.$$

(*d*) Transposing, $92x-85x=115-17$.

(*e*) Combining, $7x=98$.

(*f*) Dividing, $x=14$. *Ans.*

Test: Substituting 14 for x,

first member becomes $\frac{56-5}{17}\equiv\frac{51}{17}\equiv 3$;

second member becomes $\frac{70-1}{23}\equiv\frac{69}{23}\equiv 3$.

In practice, steps (*a*) and (*b*) are omitted. The first member of (*c*) consists of the product of the numerator $4x-5$ by the denominator 23, which is written directly, as is the second member, which is the product of the numerator $5x-1$ by the denominator 17. It is also possible to omit (*d*), by making the necessary subtractions without indicating them.

28. $\frac{4x+7}{9}=\frac{7x-5}{10}$.

29. $\frac{2x+30}{17}=\frac{3x+26}{16}$.

30. $\frac{12x+1}{19}=\frac{5x-13}{6}$.

31. $\frac{3x+7}{4}=\frac{14x+2}{11}$.

32. $\frac{x-7}{2}=\frac{3x+7}{13}$.

33. $\frac{9x+4}{29}=\frac{11x-4}{31}$.

34. $\frac{x+1}{3}=\frac{2x-4}{5}.$

35. $\frac{7x+16}{24}=\frac{6x+3}{17}.$

36. $\frac{5x-4}{4}=\frac{6x+4}{7}.$

37. $\frac{5x-1}{7}=\frac{3x+7}{8}.$

38. $\frac{5}{x+3}=\frac{2}{x-3}.$

Clearing of fractions, $(x-3)5=(x+3)2$.

39. $\frac{9}{4x+16}=\frac{10}{7x+5}.$

40. $\frac{4}{3x+11}=\frac{11}{14x+13}.$

41. $\frac{3}{x+4}=\frac{5}{2x+1}.$

42. $\frac{4}{5x-8}=\frac{7}{6x-3}.$

43. $\frac{29}{9x-25}=\frac{31}{11x-35}.$

44. $\frac{19}{12x+20}=\frac{6}{5x-7}.$

45. $\frac{2}{x-9}=\frac{13}{3x-6}.$

46. $\frac{24}{7x-8}=\frac{17}{6x-14}.$

47. $\frac{7}{5x+6}=\frac{8}{3x+15}.$

48. $\frac{17}{2x+13}=\frac{16}{3x+10}.$

49. $\frac{3x+6}{2x-9}=\frac{11}{3}.$

Clearing of fractions, $3(3x+6)=11(2x-9)$.

50. $\frac{3x}{17x+1}=\frac{4}{23}.$

51. $\frac{x-10}{x+2}=\frac{1}{5}$

52. $\frac{x-1}{6x-3}=\frac{3}{19}.$

53. $\frac{7x-6}{2x+3}=\frac{12}{5}.$

54. $\frac{3x-1}{5x-11}=\frac{5}{6}.$

55. $\frac{x+5}{3x-5}=\frac{4}{7}.$

56. $\frac{5x-24}{x+9}=\frac{12}{7}.$

57. $\frac{8x+4}{5x+4}=\frac{17}{11}.$

58. $\frac{4x+2}{9x-1}=\frac{7}{13}.$

59. $\frac{5x+2}{3x-3}=\frac{9}{4}.$

WRITTEN PROBLEMS

1. A has 34 sheep and B has 22. How many must A buy from B in order that he may have 3 times as many as B has remaining?

SOLUTION

Let x = number A needs to buy from B.
Then $34 + x$ = number A would then have,
and $22 - x$ = number B would then have.

$$34 + x = 3(22 - x).$$

2. Divide 504 into two parts, the sum of one third of one part and one fourth of the other being 138.

3. After selling $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ of his land, a man has 39 acres remaining. How many acres had he at first?

$$x - \left(\frac{x}{3} + \frac{x}{4} + \frac{x}{5}\right) = 39.$$

4. A farmer sowed 62 bushels of grain. He sowed 5 more bushels of wheat than of rye, and as many bushels of oats as of wheat and rye together. How many bushels did he sow of each?

5. A father is 48 and his son is 9. In how many years will the father's age be 4 times that of his son?

$$48 + x = 4(9 + x).$$

6. A man gives each of a number of boys 4 marbles, and he has 12 remaining. To give 5 marbles to each he would require 7 more marbles. How many marbles has he?

Let x = number of boys.
Then (a) $4x + 12$ = number of marbles,
or (b) $5x - ?$ = number of marbles.

$$(a) = (b).$$

7. What number added to both terms of the fraction $\frac{7}{13}$ makes it equal to $\frac{4}{5}$?

$$\frac{7 + x}{13 + x} = \frac{4}{5}.$$

8. A number is as much less than 39 as 5 times the number exceeds 39. Find the number.

9. A certain number is divided by 7. The sum of the divisor, the dividend, and the quotient is 127. Find the number.

Let $x =$ quotient,
and given $7 =$ divisor.
$? =$ dividend.

10. A man is 42 years old, and his daughter is 8. In how many years will the daughter's age be one third of her father's?

11. The difference between two numbers is 27, and if 6 be added to each, the greater becomes 4 times as large as the smaller. What are the numbers?

12. The sum of the ages of two brothers is 58 years, and one is 6 years older than the other. What is the age of each?

13. A certain fraction is equal to $\frac{1}{3}$. When 1 is subtracted from the numerator, and 5 is added to the denominator, its value becomes $\frac{1}{4}$. Find the fraction.

Let $\frac{x}{3x}$ represent the fraction.

14. A certain number is multiplied by 4, to the product 7 is added, the sum is divided by 5, and 6 is deducted from the quotient, leaving a remainder of 1. Find the number.

$$\frac{4x+7}{5} - 6 = 1.$$

15. A man received \$32 for a piece of work at the rate of \$3 per working day, with a deduction for board of \$1 per day, including 10 days on which he was idle. How many days did he work?

16. A saddle is worth \$135 less than a horse, and $\frac{3}{5}$ of the value of the horse is 6 times the value of the saddle. Find the value of both.

17. A man spent $2.24 for fruit, buying apples at 2¢ each, twice as many oranges at 3¢ each, and three times as many pears as oranges at 4¢ each. How many of each did he buy?

18. Two men divide their profits of $1275 in such a way as to give one $\frac{7}{8}$ as much as the other. What share does each receive?

$$x,\ \frac{7x}{8}.$$

19. After $\frac{1}{3}$ of a barrel of oil was lost through leakage, and 7 gallons were sold, the barrel was found to be half full. How many gallons did it then contain?

20. A divided with B 44 peaches in such a way that if he had given B 12 more, he would have only $\frac{2}{9}$ as many as B. How many did B receive?

21. Divide 144 into two parts that shall have the ratio of 7 to 9.

$$x,\ \frac{7x}{9}.$$

22. A grocer mixed 20 pounds of black tea costing 30¢ per pound with a quantity of green tea costing 40¢ per pound so that the mixture cost 32¢ per pound. How many pounds of green tea did he use?

23. After earning $\frac{1}{2}$ as much as he originally had, and spending $40, a man's money was $\frac{5}{6}$ of the original sum. How much had he at first?

24. A debt of $54 is paid in $2 and $5 bills, 15 in all. How many are there of each denomination?

x = number of $2 bills, $15 - x$ = number of $5 bills.

25. A tract of land containing 256 acres is so divided that the difference between the portions is 31 acres more than $\frac{1}{4}$ the number of acres in the smaller portion. How many acres are there in each?

Ratio and Proportion

Ratio is the relation between two like numbers obtained by dividing the first by the second. The sign of ratio is the colon (:), which is merely another form of the division sign.

Thus, the ratio of 10 to 7 is indicated by 10 : 7.

The ratio of 30 to 20 may be obtained by writing these numbers as a fraction, $\frac{30}{20}$, reducing the latter to lowest terms, $\frac{3}{2}$, and expressing the result 3 : 2. Any two numbers obtained by multiplying 3 and 2 by the same number will have this ratio; 24 and 16, for instance, 60 and 40, $3x$ and $2x$, etc.

Proportion is the expression of equality between two equal ratios. The sign of proportion is the double colon (: :), which is another form of the sign of equality.

Since 24 and 16 have the same ratio as 60 and 40, these four numbers form a proportion, which is written thus:

$$24 : 16 :: 60 : 40.$$

This is read "24 is to 16 as 60 is to 40."

It is equivalent to the following:

$$\frac{24}{16} = \frac{60}{40}.$$

The first and fourth terms of a proportion constitute the **extremes**; the second and third, the **means**.

The proportion $24 : 16 :: x : 40$, may be written:

$$24 \div 16 = x \div 40,$$

which is equivalent to the following:

$$\frac{24}{16} = \frac{x}{40}.$$

Clearing of fractions, this becomes

$$16x = 960.$$

An unknown number in a proportion is found by solving an equation, one member of which is the product of the extremes, and the other the product of the means.

SIGHT EXERCISES

Give the value of x:

1. $\frac{7}{14}=\frac{21}{x}$.
4. $\frac{x}{7}=\frac{25}{35}$.
7. $\frac{16}{36}=\frac{x}{9}$.
2. $\frac{5}{32}=\frac{x}{64}$.
5. $\frac{14}{45}=\frac{x}{90}$.
8. $\frac{5}{x}=\frac{15}{24}$.
3. $\frac{3}{x}=\frac{9}{27}$.
6. $\frac{3}{5}=\frac{12}{x}$.
9. $\frac{x}{13}=\frac{15}{39}$.

10. $7 \div 14 = 21 \div x$.

11. $5 : 32 = x : 64$.

12. $3 : x :: 9 : 27$.

13. $3 : 2\frac{2}{5} :: 5 : x$.

14. $x : 12 :: 3 : \frac{3}{4}$.

15. $x : \frac{1}{2} :: 2 : 7$.

WRITTEN PROBLEMS

1. Two numbers are in the ratio of 3 to 4, and their sum is 154. What are the numbers?

Suggestion. Represent the numbers by $3x$ and $4x$, respectively.

2. Two numbers are in the ratio 3 to 4. When each is increased by 10, the ratio of their respective sums is $5:6$. Find the numbers.

Method

Representing the numbers by $3x$ and $4x$, respectively, the sums, after each is increased by 10, will be $3x+10$ and $4x+10$, respectively. As the ratio of these sums is $5:6$, the following proportion results:

$$3x+10 : 4x+10 :: 5 : 6,$$

which gives rise to the equation

$$6(3x+10) = 5(4x+10).$$

Solve and test.

3. By what number must 20 and 30, respectively, be increased so that the sums will have the ratio of 3 to 4?

$$20+x : 30+x :: 3 : 4.$$

4. A father's age is to his son's as 15 to 4; the difference of their ages is 33 years; find the age of each.

5. A and B form a partnership, A investing $1000 more than B. What did each invest if A received $800 as his share of the profits and B received $600?

$$x + 1000 + x : x :: 800 + 600 : 600.$$

6. Two men enter into partnership. Of the year's profits of $600, M received $120 more than N. (*a*) How much did each receive? (*b*) If M invested $1000 more than N, how much did each invest?

One Method

After finding the answers to (*a*), N's share of the profits may be substituted in the proportion given below.

Entire capital : N's contribution : : Total profits : N's share.

Representing N's contribution by x, M's will be $x + 1000$, and the entire capital will be $2x + 1000$. The proportion now becomes

$$2x + 1000 : x :: 600 : \text{N's share}.$$

In the following, M's share must be supplied,

$$2x + 1000 : x + 1000 :: 600 : \text{M's share}.$$

7. One of two partners furnishes $1000 less than $\frac{2}{3}$ of the capital and receives $1008 as his share of the profits. The other's share is $720. What sum was invested by each if the profits are divided in the ratio of the respective investments?

8. When 23 and 29 are respectively diminished by the same number, their remainders have the ratio of 3 to 4. Find the number.

9. The ratio of a mother's age to that of her daughter is 9 : 3. In how many years will the ratio be 11 : 6?

10. A's capital was invested in business for 8 months, B's capital, which was $500 more than A's, was in the business for 6 months. Of the $1350 profits, B received $750. Assuming that the profits were divided in proportion to the capital of each multiplied by the time it was invested, find the capital of each.

Two Unknown Numbers

PRELIMINARY EXERCISES

1. If a boy pays 8 cents for a banana and two oranges, and the oranges cost 3 cents each, what is the cost of a banana?

2. When a banana and an orange cost together 5 cents and a banana and two oranges cost 8 cents, what is the cost (*a*) of an orange? (*b*) Of a banana?

3. The sum of two numbers is 48. The sum of the first and twice the second is 68. Find (*a*) the second number. (*b*) The first number.

4. In the equations:

$$(a) \qquad x + 2y = 68,$$
$$(b) \qquad x + y = 48,$$

subtract (*b*) from (*a*) member by member.

5. Given the equations:

$$(c) \qquad 3x + 5y = 54,$$
$$(d) \qquad 3x + 3y = 42.$$

Find the value of $2y$. Of y.

6. Add member by member the following equations:

$$x + y = 12,$$
$$x - y = 2.$$

What is the value of x?

7. Subtract member by member the second of the foregoing equations from the first. What is the value of y?

Two equations are said to be **simultaneous** when each of the two unknown numbers has the same value in each equation. The first step in the solution of two simultaneous equations is their combination into an equation containing only one unknown number by the process of **elimination.**

WRITTEN EXERCISES

Find the value of the unknown numbers in each of the following simultaneous equations:

1. $3x + 2y = 29$, and $x + 3y = 19$.

SOLUTION

Given $3x + 2y = 29$ (a) and $x + 3y = 19$ (b).

$$\begin{array}{lrl} & 3x + 2y = 29 & (a) \\ \text{Multiplying both members of } (b) \text{ by } 3, & 3x + 9y = 57 & (c) \\ \cline{2-2} \text{Subtracting } (a) \text{ from } (c), & 7y = 28 & \\ & y = 4. & \textit{Ans.} \end{array}$$

The value of x is found by substituting 4 for y in (b).

$$x + 12 = 19,$$
$$x = 7. \quad \textit{Ans.}$$

These values are tested by substituting 7 for x and 4 for y in (a), the first member of which then becomes $21 + 8 \equiv 29$. As 29 is the second member of (a) the values are correct.

To eliminate x, the given equations must be transformed into equivalent ones in each of which x has the same coefficient. In order that the coefficient of y may be positive in the derived equation, (a) is made the subtrahend, and its signs are changed.

NOTE. Although x is eliminated by subtracting one of the foregoing equations from the other, the process is called **elimination by addition**, the two equations being finally **added** when the signs of the subtrahend are changed.

2. $2x + 8y = 48$ and $2x + 10y = 58$.
3. $x - y = 20$ and $x + y = 72$.
4. $v + w = 15$ and $2v + 3w = 40$.
5. $2x - 4y = 22$ and $2x - 3y = 35$.
6. $2x + 7y = 69$ and $12x - 7y = 71$.
7. $5x + 3y = 77$ and $x - y = 1$.
8. $8x - 7y = 7$ and $2x + 5y = 22$.
9. $3x + 5y = 32$ and $12x - 11y = 4$.

10. $3x+2y=29$, and $2x+3y=26$.

Solution

Given $3x+2y=29$ (*a*) and $2x+3y=26$ (*b*).

$$\begin{array}{ll} \text{Multiplying both members of } (a) \text{ by 2,} & \overset{-}{6x}+\overset{-}{4y}=\overset{-}{58}\ (d). \\ \text{Multiplying both members of } (b) \text{ by 3,} & \underline{6x+9y=78\ (e).} \\ \text{Subtracting } (d) \text{ from } (e) & 5y=20. \\ & y=4.\ \textit{Ans.} \end{array}$$

Substituting this value of y in (*a*),

$$3x+8=29,\ 3x=21,\ x=7.\ \textit{Ans.}$$

The results are tested by substituting them in the other equation, (*b*).

When $x=7$ and $y=4$, the first member of (*b*) becomes

$$2\cdot7+3\cdot4\equiv14+12\equiv26.$$

As 26 is the second member, the results are correct.

11. $15x+3y=159$ and $17x-2y=164$.

Solution

$$\begin{array}{lll} \text{Given } 15x+3y=159\ (a), & \text{Multiplying } (a) \text{ by 2,} & 30x+6y=318\ (c) \\ \text{and } \quad 17x-2y=164\ (b). & \text{Multiplying } (b) \text{ by 3,} & \underline{51x-6y=492\ (d)} \\ & \text{Adding } (c) \text{ and } (d) & 81x=810 \\ & & \therefore x=10 \end{array}$$

The elimination of x would require that (*a*) be multiplied by 17 and (*b*) by 15; y is eliminated by the use of smaller numbers. As the signs of $6y$ are different, (*c*) and (*d*) are added.

Find the value of y by substituting 10 for x in (*a*) or in (*b*).

12. $5y-2z=5$ and $3y+5z=34$.

Suggestion

$$\begin{array}{lll} \text{Given } 5y-2z=\ 5\ (a), & \text{Multiplying } (a) \text{ by 3,} & \overset{-}{15y}-\overset{+}{6z}=\overset{-}{15}\ (c) \\ \text{and } \quad 3y+5z=34\ (b). & \text{Multiplying } (b) \text{ by 5,} & \underline{15y+25z=170}\ (d) \end{array}$$

As (*c*) is the smaller, subtract (*c*) from (*d*).

Or multiplying (*a*) by 5 and (*b*) by 2, then eliminate z by adding the transformed equations.

13. $10x - 3y = 75$ and $9x + 5y = 106$.

14. $3v + 2w = 17$ and $15v - 10w = 5$.

NOTE. Divide the second equation by 5.

15. $5x - y = 99$ and $x + 5y = 25$.

16. $2x - 4y = 18$ and $5x - 2y = 69$.

Divide the first equation by 2.

17. $5x - 2y = 4$ and $2x + 6y = 22$.

18. $4x - 5y = 23$ and $6x + 2y = 44$.

19. $3x + 4y = 69$ and $5x + 7y = 119$.

20. $15x = 14y + 47$ and $9x = 59 - 7y$.

By transposition, these become

$$15x - 14y = 47 \text{ and } 9x + 7y = 59.$$

21. $23x = 42 + 18y$ and $11x = 171 - 3y$.

22. $\frac{x}{3} + \frac{y}{5} = 8$ and $\frac{x}{9} - \frac{y}{10} = 1$.

Clearing of fractions, these become

$$5x + 3y = 120 \text{ and } 10x - 9y = 90.$$

23. $\frac{x}{3} - \frac{3y}{4} = 0$ and $\frac{5x}{9} + \frac{7y}{4} = 12$.

24. $6(x + 7) = 5(y + 7)$ and $3(x - 2) = y - 2$.

First perform indicated multiplications, then transpose.

25. $\frac{2x}{5} = \frac{3y}{4} - 96$ and $\frac{4x}{9} = \frac{5y}{8}$.

Clear of fractions. Transpose.

26. $\frac{x + 1}{y + 6} = \frac{3}{5}$ and $\frac{x - 7}{y + 6} = \frac{1}{15}$.

27. $5(x + 1) = 3(y + 6)$ and $8(x - 1) = 7(y - 1)$.

28. $\frac{2}{3}x + \frac{3}{4}y = 15\frac{2}{3}$ and $\frac{3}{4}x - \frac{1}{7}y = 5\frac{11}{14}$.

29. $3x = 4y$ and $5(x + 12) = 6(y + 12)$.

30. $\frac{5x + 3y}{7} = 11$ and $\frac{4x - 2y}{11} = 2$.

WRITTEN PROBLEMS

1. Ten times one number diminished by 13 times a second number gives a remainder of 41. Seven times the former diminished by 11 times the latter gives a remainder of 23. Find the numbers.

2. The difference between two numbers is 33. When each is increased by 6, the larger becomes 4 times as great as the smaller. What are the numbers?

$$x - y = 33,\ x + 6 = 4(y + 6).$$

3. In making a payment of $75, a man used 20 bills of one denomination and 7 of another; in making a payment of $52 he used 6 bills of the first denomination and 8 of the second. Find the denomination of each.

Let x = denomination of first kind; y = denomination of second kind.

Then $$20x + 7y = 75, \text{ etc.}$$

4. An employer paid $240 daily to 52 mechanics, some of whom were paid $3 per day, the others receiving $5 per day. How many men did he employ at each rate?

5. A farmer having 90 sheep and cows found after he had sold 10 of each that he then had 6 times as many of the former as of the latter. How many of each had he at first?

$$x + y = 90,\ x - 10 = 6(y - 10).$$

6. When 1 is added to the numerator of a certain fraction and 2 is added to the denominator, its value becomes $\frac{2}{3}$; and when 2 is subtracted from the numerator and 1 is subtracted from the denominator, its value becomes $\frac{1}{3}$. Find the fraction.

$$\frac{x+1}{y+2} = \frac{2}{3},\ \frac{x-2}{y-1} = \frac{1}{3}.$$

7. The sum of two numbers is 60 and their difference is 40. What are the numbers?

Solve by two unknown quantities, and also by one unknown quantity.

8. When 7 is added to each term of a certain fraction it becomes equal to $\frac{5}{6}$, and when 2 is subtracted from each term it becomes equal to $\frac{1}{3}$. What is the fraction?

9. A's age is 1 year less than $\frac{1}{2}$ of B's, and B's age is 16 years more than A's. How old is each?

10. Two pounds of sugar and 3 pounds of coffee cost 87 cents, 3 pounds of sugar and 2 pounds of coffee cost 68 cents. Find the price per pound of each.

11. If John should give James 2 marbles, the latter would then have 4 times as many as the former; if James should give John 4 marbles, the latter would have 4 times as many as the former. How many marbles has each?

12. A number is divided into two parts such that the sum of the first and $\frac{1}{2}$ the second is 56, and the second added to $\frac{1}{2}$ the first equals 58. What is the number?

13. When 1 is added to the numerator of a fraction, its value becomes $\frac{1}{4}$, when 1 is added to the denominator, its value becomes $\frac{1}{5}$. Find the fractions.

Representing the fraction by $\frac{x}{y}$, $\frac{x+1}{y}=\frac{1}{4}$, and $\frac{x}{y+1}=\frac{1}{5}$.

14. Three years from now William will be twice as old as John was 3 years ago. Three years from now John will be as old as William was three years ago. How old is each?

15. By adding 6 to the denominator of a certain fraction, its value becomes $\frac{1}{3}$, and by adding 1 to the numerator, its value becomes $\frac{1}{2}$. What is the fraction?

16. A boy having 75 peaches and oranges found when he had sold 5 of the former and 10 of the latter that he had the same number of each. How many of each did he have originally?

17. A grocer has green tea costing 40 cents per pound and black tea costing 30 cents per pound. How many pounds of each must he take to make 40 pounds of mixed tea costing 36 cents per pound?

18. A's money added to twice B's amounts to $950; B's money added to twice A's amounts to $850. How much money has each?

19. Three years ago a daughter's age was $\frac{1}{6}$ of her mother's; three years from now it will be $\frac{1}{3}$ of her mother's. What is the age of each?

20. Three times the larger of two numbers is twice their sum. Three times the smaller is 8 more than twice their difference. Find the numbers.

21. When a certain number, consisting of two digits, is divided by 4 more than the tens' digit, the quotient is 7. When the number is inverted and is divided by 4 more than the units' digit, the quotient is 6. What is the number?

When $x =$ tens' digit and $y =$ units' digit, $10x + y =$ number, and $x + 10y =$ number inverted.

22. The sum of the two digits composing a number is 11, and if 27 be added to the number the digits will be inverted. Find the number.

23. A number of two digits is equal to 7 times the sum of the digits, and when 18 is subtracted from the number, the digits are inverted. What is the number?

24. A dealer mixes 70 pounds of coffee, a portion costing 21 cents per pound and the remainder costing 28 cents per pound. How many pounds of each does he use if the mixture costs 25 cents per pound?

CHAPTER II

QUADRATIC EQUATIONS

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Symbols and Operations

EXPONENTS

To indicate the product of 4 by a, the result is written $4a$; the product of a by b is written ab; the product of a by a is written a^2. The small 2 written above the a and to its right is called an *exponent*, which indicates that a is employed twice as a factor. The expression a^2 is read "a square."

An expression consisting of one term is called a *monomial;* one consisting of two terms is called a *binomial.*

Product of Two Binomials

Multiply $a-4$ by $a-4$.

This means that from a times $(a-4)$ is subtracted 4 times $(a-4)$; the result being $a(a-4)-4(a-4)$, or $(a^2-4a)-(4a-16)$, or, removing the parentheses and simplifying,

$$a^2-8a+16. \quad \textit{Ans.}$$

The following is the general method of arranging the partial products:

Process

$$\begin{array}{ll} & a\ -4 \\ & \underline{a\ -4} \\ \text{Product by } a, & a^2-4a \\ \text{Product by } -4, & \underline{\qquad -4a+16} \\ \text{Product by } (a-4), & a^2-8a+16.\ \textit{Ans.} \end{array}$$

Placing one quantity under the other, the lower left-hand term a is first taken as a multiplier, giving the first partial product. The second term, -4, is then taken as a multiplier, giving the second partial product, the first term of the latter, $-4a$, being written in the column containing the similar term of the first partial product. The second term, $+16$, is written as shown.

Law of Signs in Multiplication

It will be observed in the foregoing example that:

$$(+a)\cdot(+a) \equiv +a^2. \qquad (+a)\cdot(-4) \equiv -4a.$$
$$(-4)\cdot(+a) \equiv -4a. \qquad (-4)\cdot(-4) \equiv +16.$$

That is, that the product of two positive terms is positive, that the product of a positive term and a negative term is negative, and that the product of two negative terms is positive. This is expressed in the following law:

In multiplication like signs give $+$; *unlike signs give* $-$.

WRITTEN EXERCISES

Simplify the following:

1. $(m+5)(m+6)$. 2. $(2a-3)(a-3)$. 3. $(2x+7)(3x-7)$.

Process	Process	Process
$m+5$	$2a\ -3$	$2x\ +7$
$\underline{m+6}$	$\underline{\ \ a\ -3}$	$\underline{3x\ -7}$
m^2+5m	$2a^2-3a$	$6x^2+21x$
$\underline{\qquad 6m+30}$	$\underline{\qquad -6a+9}$	$\underline{\qquad -14x-49}$
combine	combine	combine

4. $(m-7)(m+6)$. 6. $(a-7)(a-7)$.

5. $(x+4)(x-4)$. 7. $(v-9)(v-10)$.

Simple Equations Quadratic in Form

An equation in which the unknown number has no exponent is called a **simple** equation, or a **linear** equation, or an equation of the **first degree.** A **quadratic** equation is one that contains the square of the unknown number; for which reason it is also called an equation of the **second degree.**

Equations that contain a term of the second degree that is removable by the simplification of the equation, although apparently of the second degree, are really simple equations, and are solved as such.

WRITTEN EXERCISES

Solve the following:

1. $(m+3)(m-7)=(m-1)(m-6)$.

Solution

Given $(m+3)(m-7)=(m-1)(m-6)$.

Performing the indicated multiplications,

$$m^2-4m-21=m^2-7m+6.$$

Transposing and combining,

$$3m=27.$$

Dividing by 3, $m=9$. *Ans.*

Test

First member $(9+3)(9-7)\equiv 12\cdot 2\equiv 24$.

Second member $(9-1)(9-6)\equiv 8\cdot 3\equiv 24$.

2. $(x-3)(x-4)=(x-7)(x+4)$.

3. $(2y+2)(y+2)=(2y-1)(y+6)$.

4. $(2z+4)(2z-4)=(2z-1)(2z-2)$.

5. $(3a-9)(a+1)=(3a-6)(a-1)$.

6. $\frac{3v-9}{3v-6}=\frac{v-1}{v+1}$. Clearing of fractions, $(3v-9)(v+1)=(3v-6)(v-1)$.

7. $\frac{4w-12}{w-1}=\frac{4w-8}{w+1}$.

8. $\frac{x+3}{x-5}=\frac{x-2}{x-6}$.

9. $\frac{x+4}{x+1}=\frac{x+6}{x+2}$.

10. $\frac{x-1}{x-7}=\frac{x+3}{x-6}$.

WRITTEN PROBLEMS

1. The hypotenuse of a right triangle is 2 feet longer than the base, and the perpendicular is 8 feet. Find the length of the base.

SOLUTION

Let $x =$ length of base.

Then $x + 2 =$ length of hypotenuse,

and since $(\text{base})^2 + (\text{perpendicular})^2 = (\text{hypotenuse})^2$,

$$x^2 + 8^2 = (x + 2)^2,$$

that is, $x^2 + 64 = x^2 + 4x + 4$.

Complete the solution, and test the result.

2. The difference of two numbers is 1, and the difference of their squares is 25. Find the numbers.

$$(x + 1)^2 - x^2 = 25.$$

3. The area of a square field is the same as that of a rectangular field having one of its sides 20 rods less than a side of the square, the adjacent side being 30 rods greater than a side of the square. Find (*a*) the side of the square. (*b*) The dimensions of the rectangle.

$x =$ side of square, $x - 20$ and $x + 30 =$ dimensions of rectangle.

4. One side of a rectangular plot is 7 yards shorter than the adjacent one. Its area is equal to that of a square having a side 3 yards longer than one of the shorter sides of the rectangle. Find (*a*) the dimensions of the rectangle. (*b*) The area of the rectangle.

Short side $= x$; long side $= x + 7$; side of square $= x + 3$; $x(x + 7) = (x + 3)^2$, etc.

5. The difference of the squares of two consecutive numbers is 51. Find the numbers.

$$x, x + 1.$$

6. The difference between the square of 3 more than a number and the square of 7 more than the same number is 256. Find the number.

7. The difference between the hypotenuse of a right triangle and one of its sides is 18 yards. The other side measures 24 yards. Find the length of the hypotenuse.

8. The difference between two numbers is 6, and the difference between their squares is 72. What are the numbers?

$$x,\ x+6.$$

9. The quotient of 3 less than a number divided by 1 less than the number equals the quotient of 2 less than the number divided by 1 more than the number. Find the number.

10. The product of 3 less than a number multiplied by 1 more than the number is equal to the product of 2 less than the number multiplied by 1 less than the number.

11. What number must be added to each of the numbers, 3, 13, 8, and 23, so that the resulting sums may form a proportion?

$$3+x : 13+x :: 8+x : 23+x.$$

12. What number added to each term of the two fractions $\frac{3}{18}$ and $\frac{8}{28}$ will make the resulting fractions equal?

13. What number must be subtracted from each of the numbers, 15, 19, 21, and 27, so that the remainders may form a proportion?

14. When each term of the fractions $\frac{5}{7}$ and $\frac{6}{9}$ is diminished by a certain number, the resulting fractions are equal. Find the value of the latter in lowest terms.

15. From what number must 18, 14, 12, and 6 be subtracted in order that the resulting remainders may form a proportion?

16. A farmer has two square fields, the perimeter of one being 4 rods greater than that of the other and the difference in their areas being 81 square rods. What is the perimeter of each?

When x represents a side of the smaller field, what represents a side of the larger one?

Incomplete Quadratic Equations

An **incomplete quadratic equation** is one in which only the square of the unknown number appears; a **complete quadratic equation** being one that contains both the first power and the second power of the unknown number. An incomplete quadratic equation is also called a **pure quadratic.**

Since 16 is the product of -4 by -4, as well as the product of $+4$ by $+4$, the algebraic square root of 16 is either $+4$ or -4.

The sign $\sqrt{\ }$, called the radical sign, indicates the square root of the quantity it precedes. Thus, $\sqrt{16} \equiv \pm 4$, the combination of the $+$ and the $-$ sign prefixed to 4, denotes that this number may be either positive or negative.

SIGHT EXERCISES

Give the values of x:

1. $x^2 - 9 = 16.$
2. $2x^2 + 1 = 19.$
3. $3x^2 - 1 = 74.$
4. $2x^2 = x^2 + 16.$
5. $4x^2 = 18 + 2x^2.$
6. $2x^2 + x^2 = 50 + 25.$
7. $3x^2 + 25 = 50 + 2x^2.$
8. $40 + 2x^2 = x^2 + 49.$
9. $x^2 + 50 = 2x^2 + 1.$
10. $3x^2 + 3 = 2x^2 + 39.$

WRITTEN EXERCISES

1. $7x^2 + 10 = 353.$
2. $x^2 + 1 = \frac{x^2 + 16}{4}.$
3. $\frac{x^2 - 1}{8} = 10.$
4. $(x + 4)(x - 4) = 33.$
5. $\frac{x + 3}{4} = \frac{18}{x - 3}.$
6. $5x^2 - 10 = 115.$
7. $2x^2 - \frac{x^2}{3} = 15.$
8. $\frac{2x^2 - 2}{7} = 10.$
9. $(2x - 1)(2x + 1) = 35.$
10. $\frac{2x - 5}{13} = \frac{23}{2x + 5}.$

NOTE. While x has two values in each of the foregoing exercises, only the positive value need be given in the answers to the following problems.

WRITTEN PROBLEMS

1. The base of a right-angled triangle is $\frac{5}{12}$ of the perpendicular, and the hypotenuse measures 13 yards. Find (*a*) the perpendicular, (*b*) the base, (*c*) the area.

$x =$ perpendicular. $\frac{5x}{12} =$ base. $13^2 = x^2 + \left(\frac{5x}{12}\right)^2.$

2. One side of a rectangular field is twice as long as the other, and the area of the field is 3200 square rods. Find the dimensions of the field.

3. Two thirds of a number multiplied by three fourths of the number is 72. What is the number?

4. The product of two numbers in the ratio of 7 to 9 is 252. Find the numbers.

Let $7x =$ one number; then $9x =$ other number.

5. The side of one square field is twice that of another, and their combined area is 500 square rods. What is the side of each?

6. The product of two numbers is 343, and their quotient is 7. Find the numbers.

Let $x =$ less. Then $7x =$ greater.

7. A man has three square fields the side of the first being 4 rods shorter than the side of the second, and the side of the third being 4 rods longer than the side of the second. The combined area of the first and the third is 176 square rods greater than the area of the second. What is the length of the side of each?

$$(x-4)^2 + (x+4)^2 = x^2 + 176.$$

8. The product of 5 more than a number by 5 less than the same number is 375. Find the number.

9. Given the number of square feet in the surface of a circle as $3\frac{1}{7}$ times the square of the number of linear feet in the radius, find the radius of a circle whose area is 154 square feet.

Complete Quadratic Equations

SQUARING A BINOMIAL

Multiply $(x+5)$ by $(x+5)$.

$$\begin{array}{lrrr}
 & x & +5 & \\
 & x & +5 & \\ \cline{2-4}
 & x^2 & +5x & \\
 & & 5x & +25 \\ \cline{2-4}
\textit{Ans.} & x^2 & +10x & +25
\end{array}$$

The square of the sum of two quantities is equal to the square of the first + twice the product of the first by the second + the square of the second.

Multiply $(x-5)$ by $(x-5)$.

The square of the difference of two quantities is equal to the square of the first − twice the product of the first by the second + the square of the second.

$$\begin{array}{lrrr}
 & x & -5 & \\
 & x & -5 & \\ \cline{2-4}
 & x^2 & -5x & \\
 & & -5x & +25 \\ \cline{2-4}
\textit{Ans.} & x^2 & -10x & +25
\end{array}$$

Thus, $(m+3)^2 \equiv m^2+6m+9$, $(v-10)^2 \equiv v^2-20v+100$.

SIGHT EXERCISES

Square the following binomials:

1. $(m+6)^2$.
2. $(b+5)^2$.
3. $(v-2)^2$.
4. $(d+8)^2$.
5. $(x-4)^2$.
6. $(y+4)^2$.
7. $(w-11)^2$.
8. $(e-9)^2$.
9. $(a+2)^2$.
10. $(z-5)^2$.
11. $(c-7)^2$.
12. $(f+10)^2$.

Note. **An expression of three terms is called a trinomial.**

Give the square root of the following:

1. a^2+2a+1.
2. $b^2+8b+16$.
3. $c^2+10c+25$.
4. $d^2-8d+16$.
5. $m^2+14m+49$.
6. a^2-2a+1.
7. f^2-6f+9.
8. $u^2-10u+25$.
9. v^2-4v+4.
10. $e^2+18e+81$.
11. $w^2+12w+36$.
12. x^2+6x+9.
13. $y^2-12y+36$.
14. z^2+4z+4.
15. $n^2-16n+64$.

The formation of a trinomial that is a perfect square by the addition of a number is called **completing the square.**

SIGHT EXERCISES

Make the following expressions perfect squares:

1. $x^2+14x+?$
2. $x^2-2x+?$
3. $x^2+16x+?$
4. $x^2-20x+?$
5. $x^2+40x+?$
6. $x^2+18x+?$
7. $x^2-14x+?$
8. $x^2+24x+?$
9. $x^2-40x+?$
10. $x^2+20x+?$
11. $x^2-16x+?$
12. $x^2+22x+?$
13. $x^2-18x+?$
14. $x^2+24x+?$
15. $x^2-60x+?$

Find the roots of the equation: $x^2-6x=7$.

General Method

Given $x^2-6x=7.$

(*a*) Adding the square of ½ of -6 to each member,

$$x^2-6x+9=7+9,$$

(*b*) or, $x^2-6x+9=16.$

(*c*) Extracting the square root of both members,

$$x-3=\pm 4.$$

(*d*) Transposing, $x=4+3$, or $-4+3$.

(*e*) Combining, $x=7$, or -1. *Ans.*

Test : When $x=7$, the first member becomes

$$7^2-6\cdot 7\equiv 49-42\equiv 7.$$

When $x=-1$, the first member becomes

$$(-1)^2-6(-1)\equiv 1+6\equiv 7.$$

The first member of the given equation is made a complete square by adding the square of ½ of the coefficient of x; that is, the square of -3, or 9. To preserve the equality, 9 is also added to the second member. In (*c*) both values of $\sqrt{16}$ are given in the second member. In (*d*) the first value of x uses the positive square root of 16, and the second value uses the negative square root.

The answers obtained in the solution of an equation are called the **roots** of an equation. A linear equation has **one** root; a quadratic equation has **two** roots. When the latter is an incomplete quadratic equation, the two roots are numerically the same but have opposite signs; when it is a complete quadratic equation, the roots are numerically different.

WRITTEN EXERCISES

Find the two roots of each of the following:

1. $x^2+2x=3$.
2. $x^2+2x=24$.
3. $x^2-2x=8$.
4. $x^2-2x=35$.
5. $x^2-2x=63$.
6. $x^2-4x=5$.
7. $x^2+4x=21$.
8. $x^2-4x=32$.
9. $x^2-4x=45$.
10. $x^2+6x=7$.
11. $x^2-6x=16$.
12. $x^2+8x=20$.
13. $x^2-8x=33$.
14. $x^2+10x=-9$.
15. $x^2-10x=11$.
16. $x^2+12x=28$.
17. $x^2-12x=13$.
18. $x^2+20x=300$.
19. $x^2-30x=-125$.
20. $x^2+40x=-300$.
21. $10x^2+60x=-80$.

Dividing both members by 10, $x^2+6x=-8$.

22. $9x^2+36x=288$.
23. $7x^2-84x=-189$.
24. $8x^2-48x=440$.
25. $4x^2+32x=192$.
26. $8x^2-48x=128$.
27. $6x^2+36x=240$.
28. $3x^2-18x=165$.
29. $2x^2-36x=-154$.
30. $x^2-9x=-14$.

SOLUTION

Given $x^2-9x=-14$.

Adding the square of $\frac{1}{2}$ coefficient of x,

$$x^2-9x+\left(\tfrac{9}{2}\right)^2=-14+\tfrac{81}{4}$$

$$=\frac{-56+81}{4}=\frac{25}{4}$$

Extracting the square root $x-\frac{9}{2}=\pm\frac{5}{2}$.

Transposing, $x=\frac{5}{2}+\frac{9}{2}$ or $-\frac{5}{2}+\frac{9}{2}$

$=\frac{14}{2}$ or $\frac{4}{2}$

$=7$ or 2. *Ans.*

Test: When $x=7$, $x^2-9x\equiv 7^2-9\cdot 7\equiv 49-63\equiv -14$.

When $x=2$, $x^2-9x\equiv 2^2-9\cdot 2\equiv 4-18\equiv -14$.

31. $x^2 - 5x - 24 = 0.$

32. $x^2 - x = 20.$

33. $x^2 - 3x = 18.$

34. $x^2 - 13x = -40.$

35. $10x^2 - 70x = -60.$

36. $2x^2 - 30x = -112.$

37. $9x^2 - 9x = 108.$

38. $4x^2 - 52x = -168.$

39. $76 - 15x = 3x^2 + 4.$

Solution

Given $76 - 15x = 3x^2 + 4.$

Transposing, and changing signs,

$$3x^2 + 15x = -4 + 76 = 72.$$

Dividing by coefficient of x^2,

$$x^2 + 5x = 24.$$

Solve and test.

Note. The first term of the first member should, for the present, be x^2, positive and without a coefficient. The second member should consist of but one term, the known number. In the following equations make the necessary transpositions, divisions, and changes of signs before completing the square. Test the two roots by substituting each in the original equation.

40. $x^2 = 5x^2 + 48 - 28x.$

41. $x^2 = 560 + 11x^2 + 150x.$

42. $1 + x = 2x^2 - 9x + 9.$

43. $3x - 18 + 3x^2 = 0.$

44. $8x - 145 = 15 + 8x^2.$

45. $37x + 50 = 2x - 5x^2.$

46. $245 - 11x^2 = 55x - 19.$

47. $13x - 504 = 4x - 9x^2.$

48. $10x^2 + 21x - 126 = 3x^2.$

49. $109 + 6x^2 - 54x = 1.$

50. $x^2 = 4x^2 + 14x - 5.$

Solution

Given $x^2 = 4x^2 + 14x - 5.$

Transposing and changing signs, $3x^2 + 14x = 5.$

Dividing by 3, $x^2 + \frac{14x}{3} = \frac{5}{3}.$

Completing the square $x^2 + \frac{14x}{3} + \left(\frac{7}{3}\right)^2 = \frac{5}{3} + \frac{49}{9}$

$$= \frac{15 + 49}{9} = \frac{64}{9}, \textit{ etc.}$$

51. $8x^2 - 6x - 35 = 0.$

52. $3x^2 - 5x - 12 = 0.$

53. $2x^2 - 11x + 12 = 0.$

54. $4x^2 + 9x - 28 = 0.$

55. $2x^2 + 1 = 41x + 64 + 8x^2.$

Solution

Given $2x^2 + 1 = 41x + 64 + 8x^2.$

Transposing and changing signs, $6x^2 + 41x = -63.$

Dividing by 6, $x^2 + \frac{41x}{6} = -\frac{63}{6}.$

Adding square of $\frac{1}{2}$ of $\frac{41}{6}$, $x^2 + \frac{41x}{6} + \frac{1681}{144} = -\frac{63}{6} + \frac{1681}{144}$

$= \frac{-1512 + 1681}{144}$

$= \frac{169}{144}.$

Extracting the square root, $x + \frac{41}{12} = \pm \frac{13}{12}.$

Transposing, $x = \frac{13}{12} - \frac{41}{12} = -\frac{28}{12} = -2\frac{1}{3}.$ *Ans.*

$x = -\frac{13}{12} - \frac{41}{12} = -\frac{54}{12} = -4\frac{1}{2}.$ *Ans.*

Test the correctness of both roots by substituting each, successively, in the original equation.

56. $2x^2 + x - 3 = 0.$

57. $2x^2 + 3x + 1 = 0.$

58. $2x^2 - 13x = -20.$

59. $3x^2 - 23x = -14.$

60. $3x^2 + 8x + 5 = 0.$

61. $3x^2 + 5x - 12 = 0.$

62. $3x^2 + 5x - 2 = 0.$

63. $3x^2 - 8x + 4 = 0.$

64. $2x^2 - 15x = -18.$

65. $2x^2 - 17x + 8 = 0.$

Perform the indicated multiplications, and solve:

66. $(x-2)(x-4) = 0.$

67. $(x-3)(x-5) = 0.$

68. $(x-1)(x-3) = 0.$

69. $(x-4)(x-6) = 0.$

70. $(x+1)(x+3) = 0.$

71. $(x+7)(x-5) = 0.$

72. $(x-5)(x+7) = 0.$

73. $(x+4)(x-6) = 0.$

74. $(2x-2)(x-2) = 0.$

75. $(2x-4)(x-4) = 0.$

76. $(3x-3)(x-3) = 0.$

77. $(3x-6)(x-4) = 0.$

78. $(2x-1)(x-1) = 0.$

79. $(2x-1)(x-3) = 0.$

80. $(3x-1)(x-2) = 0.$

81. $(4x-1)(x-1) = 0.$

WRITTEN PROBLEMS

Note. Give only positive numbers in answers.

1. The difference of two numbers is 8 and their product is 48. Find the numbers.

$$x(x+8)=48.$$

2. Find two numbers whose sum is 14 and whose product is 48.

$$x(14-x).$$

3. The sum of the squares of two consecutive numbers is 61. What are the numbers?

$$x^2+(x+1)^2.$$

4. Find two consecutive numbers whose product is 156.

5. If the product of two consecutive even numbers is 168, what are the numbers?

Take x and $x+2$, or $x-1$ and $x+1$.

6. Find two consecutive odd numbers having 202 for the sum of their squares.

7. The product of two consecutive odd numbers is 1599. Find the numbers.

8. Find the dimensions of a rectangle having an area of 575 square rods and having one side 2 rods longer than the adjacent side.

9. The perimeter of a rectangle is 84 yards and its area is 440 square yards. What are its dimensions?

The sum of the dimensions is 42 yards, one half the perimeter.

10. The base of a right triangle is x feet, the perpendicular is $(x-17)$ feet, the hypotenuse is 25 feet. Find (*a*) the length of the base, (*b*) the length of the perpendicular, (*c*) the perimeter of the triangle, and (*d*) its area.

11. The area of a right triangle is 240 square yards; its base is 14 yards longer than its perpendicular. Find the length of each of the three sides.

$$\frac{x(x+14)}{2}=240.$$

12. The sum of the lengths of the base and the perpendicular of a right triangle is 35 feet and the length of the hypotenuse is 25 feet. Find (*a*) the dimensions and (*b*) the area.

$$x^2 + (35 - x)^2 = 25^2.$$

13. The perimeter of a right triangle is 56 yards. If the hypotenuse measures 25 yards, (*a*) what are the lengths of the other sides? (*b*) Find the area of the triangle.

14. Find two numbers whose difference is 6 and whose product is 135.

15. The difference of two numbers is 4 and the sum of their squares is 250. What are the numbers?

16. The entire surface of a square prism is 190 square inches; its height is 7 inches; each side of the base is x inches. Find the value of x.

Area of each base $= x^2$; area of each of the 4 convex faces is $7x$. Entire surface $= 2x^2 + 28x$.

17. The sum of the squares of three consecutive numbers is 434. What are the numbers?

18. A man has two fields; one a square and the other a rectangle. The dimensions of the latter are 4 rods and 6 rods, respectively, longer than a side of the square; and the area of the rectangle is double that of the square. Find the dimensions of each field.

19. A plot of ground 20 yards square is surrounded by a path having an area of 84 square yards. How wide is the path?

Area of plot in square feet $= 20 \times 20$. Area of plot including the path $(20 + 2x)(20 + 2x)$. Area of path $(20 + 2x)^2 - 20^2$.

20. A plot of ground is 20 yards square including a path. If the area of the path is 39 square yards, what is its width?

CHAPTER III

REVIEWS AND APPLICATIONS

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Algebraic Signs

Every number must either be equal to another, greater than another, or less than another. To indicate these relations, the following signs are employed:

$$=,\ >,\ <,$$

which are read "is equal to," "is greater than," "is less than."

EXAMPLES: $6 = 6,\ 7 > 6,\ 6 < 7.$

To indicate the identity of two algebraic numbers, the sign $\equiv$ is occasionally used.

Thus, the result of multiplying $(x - 4)$ by $(x - 6)$ may be written as an identity,

$$(x - 4)(x - 6) \equiv x^2 - 10x + 24,$$

this being true for all values of x. In the following equation

$$(x - 4)(x - 6) = 15,$$

x has but two values, 9 and 1, which are obtained by solving the equation.

SIGHT EXERCISES

State which of the three signs must be employed between the following expressions:

1. $10+5$ and 4×4.
2. $\frac{7}{8}$ and $\frac{7\times 3}{8\times 3}$.
3. $\sqrt{25}$ and $\sqrt{16}+\sqrt{9}$.
4. $\frac{3}{4}$ and $\frac{3+1}{4+1}$.
5. $\frac{3}{4}$ and $\frac{3-1}{4-1}$.
6. $23-8$ and 7×2.
7. $\frac{9}{12}$ and $\frac{9\div 3}{12\div 3}$.
8. $(3+2)^2$ and 3^2+2^2.
9. $\frac{7}{4}$ and $\frac{7+1}{4+1}$.
10. $\frac{7}{4}$ and $\frac{7-1}{4-1}$.

State in each such case what quantity must be placed in the first member to make the two members identical.

SIGNS OF OPERATIONS

The fundamental operations by which two numbers are combined to form a third, are indicated, and the terms employed are shown in the following table.

NAMES OF OPERATIONS	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
Examples	$12+4\equiv 16$	$12-4\equiv 8$	$12\times 4\equiv 48$	$12\div 4\equiv 3$
The number 12 is called	Addend	Minuend	Factor	Dividend
The number 4 is called	Addend	Subtrahend	Factor	Divisor
Result is called	Sum	Remainder	Product	Quotient

As algebra deals only with abstract numbers, the expression "multiplied by" is replaced by the shorter one "times," and both "multiplicand" and "multiplier" are replaced by "factor."

SIGNS OF AGGREGATION

To indicate by minus signs the difference between 25 and $10 - 5$, it is necessary to inclose the last two quantities in a parenthesis, thus:

$$25 - (10 - 5),$$

which means that after the operation indicated by the sign is performed upon the quantities within the parenthesis, the latter may be removed; thus:

$$25 - (10 - 5) \equiv 25 - 5,$$

before the remaining operation is performed.

In the same way, the product of 5 times the sum of 12 and 8 is indicated thus:

$$5 \times (12 + 8),$$

which may be also written without the multiplication sign, thus:

$$5(12 + 8).$$

The division by 3 of the sum of 36 and 6 is indicated thus:

$$(36 + 6) \div 3.$$

SIGHT EXERCISES

Give answers:

1. $25 - (10 - 5)$.
2. $(25 - 10) - 5$.
3. $(5 \times 12) + 8$.
4. $5 \times (12 + 8)$.
5. $25 - 10 - 5$.
6. $25 - 10 + 5$.
7. $(36 \div 6) \div 3$.
8. $36 \div (6 \div 3)$.
9. $10 \times (4 \times 2)$.
10. $(10 \times 4) \times 2$.
11. $(10 \times 4 \times 2) \div 8$.
12. $10 \times (8 \div 2) \div 4$.
13. $10 \times (3 + 1) - 8$.
14. $20 \div (4 + 1) - 2$.
15. $20 \times (4 + 1) - 2$.
16. $20 \times (4 + 1 - 2)$.
17. $(8 - 6) \times 4 - 2$.
18. $(8 - 6) \times (4 - 2)$.
19. $29 - (3 \times 2) - 2$.
20. $(9 \div 3) \times (2 \div 2)$.

ORDER OF OPERATIONS

Although a parenthesis may be employed to prevent ambiguity in expressions similar to the foregoing, there are certain established rules that determine the order in which a series of indicated operations should be performed.

First, when all the signs of an expression containing no parentheses are + and −, or × and ÷, the operations are to be performed in the order in which they are written.

SIGHT EXERCISES

Give answers:

1. $10 - 2 - 5 - 2$.
2. $10 - 2 + 5 - 2$.
3. $10 + 2 - 5 + 2$.
4. $10 + 2 + 5 - 2$.
5. $10 + 2 + 5 + 2$.
6. $10 \times 2 \times 5 \times 2$.
7. $10 \div 2 \times 5 \div 2$.
8. $10 \times 2 \div 5 \times 2$.
9. $10 \div 2 \div 5 \div 2$.
10. $10 \div 2 \div 5 \times 2$.

Secondly, when the expression contains × or ÷ signs together with + or − signs, the operations indicated by the × or ÷ signs must first be performed.

Thus,

$$\begin{aligned} 40 \div 2 \div 5 - 2 &\equiv 20 \div 5 - 2 \equiv 4 - 2. \\ 40 \div 2 \times 5 + 2 &\equiv 20 \times 5 + 2 \equiv 100 + 2. \\ 40 + 40 \div 5 \div 2 &\equiv 40 + 8 \div 2 \equiv 40 + 4. \\ 40 + 2 \times 5 - 2 &\equiv 40 + 10 - 2. \end{aligned}$$

SIGHT EXERCISES

Give answers:

1. $18 + 6 \div 3$.
2. $18 - (8 + 3)$.
3. $20 \div 5 \times 2$.
4. $20 \times 5 \div 2$.
5. $7 + 17 - 14$.
6. $20 \div (10 \div 2)$.
7. $20 \div 10 \div 2$.
8. $20 \div 10 \times 2$.
9. $5 \times 18 \div 9$.
10. $5 \times (18 \div 9)$.
11. $30 + 3 \times (5 - 2)$.
12. $30 + 3 \times 5 - 2$.
13. $30 \div 3 + 5 \times 2$.
14. $30 - 3 \div (5 - 2)$.

When it is necessary to indicate, for instance, that the difference of 5 and 2 is subtracted from 11 and the remainder is multiplied by 6, a second mark of aggregation is employed, the bracket, [ ]; thus:

$$6 \times [11 - (5 - 2)].$$

To obtain the result, the parenthesis is generally removed first, then the bracket. Thus:

$$6 \times [11 - (5 - 2)] \equiv 6 \times [11 - 3] \equiv 6 \times 8 \equiv 48.$$

When an additional mark of aggregation is required outside the bracket, a brace { } may be employed.

A straight line, called a vinculum, placed above quantities to be combined, is sometimes used, especially in arithmetical combinations. Thus: $\overline{18 + 2} \times 5 \equiv 20 \times 5$.

Beginning at the inmost:

$$\begin{aligned} 100 - \{100 - [20 - (10 + 5)]\} &\equiv 100 - \{100 - [20 - 15]\} \\ &\equiv 100 - \{100 - 5\} \\ &\equiv 100 - 95 \\ &\equiv 5 \quad \textit{Ans.} \end{aligned}$$

WRITTEN EXERCISES

Find answers:

1. $50 - [40 - 3(4 + 8)]$.
2. $(2 + 4)[4(3 + 7) - 5 \times 6]$.
3. $[4(2 + 3) - (9 - 4)](5 - 3)$.
4. $\{6[14 - (9 - 3)]\} \div 24$.
5. $[(3 + 13 - 4)5 - 2 \times 3] \div 9$.
6. $4\{20 + [19 - (8 + 9)]\}$.
7. $(3 + 9 \times 3 + 6 \div 2) \div (9 - 5)$.
8. $\overline{5 + 8 \div 2 \times 4} \div \overline{8 - 6 + 1}$.
9. $80 \div \{89 - [9 + 80 \div (4 \div 2)]\}$.
10. $(45 + 9 \times 5) \div (2 + 3 + 4)$.

NUMERICAL APPLICATIONS

Some numerical operations are facilitated by first changing a given number to one more readily handled and then making the proper modification of this result.

Thus: $187+99=(187+100)-1.$
$325+69=(325+60)+9.$
$596+49=(600+49)-4.$
$187-99=(187-100)+1.$
$472-88=(472-80)-8.$

SIGHT EXERCISES

Give answers:

1. $187+99.$
2. $375+98.$
3. $563+99.$
4. $598+177.$
5. $499+183.$
6. $397+179.$
7. $473-99.$
8. $786-99.$
9. $581-93.$
10. $576-199.$
11. $645-86.$
12. $732-399.$

$$63\times 99\equiv 63(100-1)\equiv 6300-63.$$
$$24\times 19\tfrac{7}{8}\equiv 24(20-\tfrac{1}{8})\equiv 480-\tfrac{1}{8}\text{ of }24.$$
$$36\times 9\tfrac{3}{4}\equiv 36(10-\tfrac{1}{4})\equiv 360-\tfrac{1}{4}\text{ of }36.$$

Give products:

1. $63\times 99.$
2. $99\times 93.$
3. $87\times 99.$
4. $88\times 99\tfrac{1}{2}.$
5. $88\times 99\tfrac{3}{4}.$
6. $88\times 99\tfrac{7}{8}.$
7. $24\times 19\tfrac{7}{8}.$
8. $24\times 19\tfrac{5}{6}.$
9. $24\times 19\tfrac{3}{4}.$
10. $19\tfrac{7}{8}\times 32.$
11. $19\tfrac{3}{4}\times 32.$
12. $19\tfrac{15}{16}\times 32.$

$$85\times 25\equiv(85\div 4)\times 100\equiv 21\tfrac{1}{4}\times 100\equiv 2125.$$
$$65\times 33\tfrac{1}{3}\equiv(65\div 3)\times 100\equiv 21\tfrac{2}{3}\times 100\equiv 2166\tfrac{2}{3}.$$
$$87\times 12\tfrac{1}{2}\equiv(87\div 8)\times 100\equiv 10\tfrac{7}{8}\times 100\equiv 1087\tfrac{1}{2}.$$

The usual method of multiplication by an aliquot part of 100 is to first multiply by 100 and then to divide. It is shorter to divide first, considering the quotient as hundreds, the fraction, if any, being changed to its product by 100.

13. $87\times 25.$
14. $49\times 25.$
15. $27\times 25.$
16. $96\times 33\tfrac{1}{3}.$
17. $38\times 33\tfrac{1}{3}.$
18. $67\times 33\tfrac{1}{3}.$
19. $85\times 12\tfrac{1}{2}.$
20. $47\times 12\tfrac{1}{2}.$
21. $28\times 12\tfrac{1}{2}.$
22. $67\times 16\tfrac{2}{3}.$
23. $29\times 16\tfrac{2}{3}.$
24. $55\times 16\tfrac{2}{3}.$

To obtain the product of two numbers between 11 and 19, add to one number the units of the other, consider the sum as tens, and increase this result by the product of their units.

Thus, $18 \cdot 17 \equiv (18 + 7)$ tens $+ 8 \cdot 7 \equiv 250 + 56 \equiv 306$.

That is,
$$\begin{aligned} 18(10+7) &\equiv 18\cdot 10 + 18\cdot 7. \\ &\equiv 18\cdot 10 + (10+8)7. \\ &\equiv (18\cdot 10 + 10\cdot 7) + 8\cdot 7. \\ &\equiv (18+7)10 + 8\cdot 7. \end{aligned}$$

Thus

$15 \cdot 19 \equiv (15 + 9)$ tens $+ 5 \cdot 9$. $16 \cdot 16 \equiv (16 + 6)$ tens $+ 6 \cdot 6$.

25. 13×13. **28.** 16×16. **31.** 19×19. **34.** 18×17.
26. 14×14. **29.** 17×17. **32.** 17×13. **35.** 15×16.
27. 15×15. **30.** 18×18. **33.** 16×14. **36.** 14×19.

$$103 \times 108 = (103 + 8) \text{ hundreds} + 8 \cdot 3.$$

37. 103×108. **40.** 106×107. **43.** 102×108.
38. 104×104. **41.** 108×109. **44.** 104×106.
39. 105×105. **42.** 103×107. **45.** 109×109.

The product of two mixed numbers having the same integer may be obtained by increasing one of them by the fraction in the other, multiplying the sum by the integer, and adding the product of the fractions.

Thus: $3\frac{1}{2} \cdot 3\frac{1}{4} \equiv (3\frac{1}{2} + \frac{1}{4}) \cdot 3 + \frac{1}{2} \cdot \frac{1}{4}$.

When the sum of the fractions is 1, this method shortens the work:

$$4\tfrac{1}{2} \cdot 4\tfrac{1}{2} \equiv (4\tfrac{1}{2} + \tfrac{1}{2}) \cdot 4 + \tfrac{1}{2} \cdot \tfrac{1}{4} \equiv 5 \cdot 4 + \tfrac{1}{2} \cdot \tfrac{1}{4}.$$
$$5\tfrac{1}{3} \cdot 5\tfrac{2}{3} \equiv (5\tfrac{1}{3} + \tfrac{2}{3}) \cdot 5 + \tfrac{1}{3} \cdot \tfrac{2}{3} \equiv 6 \cdot 5 + \tfrac{1}{3} \cdot \tfrac{2}{3}.$$
$$6\tfrac{1}{4} \cdot 6\tfrac{3}{4} \equiv (6\tfrac{1}{4} + \tfrac{3}{4}) \cdot 6 + \tfrac{1}{4} \cdot \tfrac{3}{4} \equiv 7 \cdot 6 + \tfrac{1}{4} \cdot \tfrac{3}{4}.$$

46. $5\frac{1}{2} \times 5\frac{1}{2}$. **49.** $3\frac{1}{3} \times 3\frac{2}{3}$. **52.** $1\frac{1}{4} \times 1\frac{3}{4}$. **55.** $4\frac{3}{10} \times 4\frac{7}{10}$.
47. $6\frac{1}{2} \times 6\frac{1}{2}$. **50.** $8\frac{1}{3} \times 8\frac{2}{3}$. **53.** $3\frac{3}{4} \times 3\frac{1}{4}$. **56.** $6\frac{1}{10} \times 6\frac{9}{10}$.
48. $9\frac{1}{2} \times 9\frac{1}{2}$. **51.** $7\frac{2}{3} \times 7\frac{1}{3}$. **54.** $5\frac{1}{4} \times 5\frac{3}{4}$. **57.** $8\frac{3}{5} \times 8\frac{2}{5}$.

The same method is applicable to mixed decimals:

58. 4.5×4.5.
59. 5.5×5.5.
60. 6.5×6.5.
61. 4.3×4.7.
62. 4.4×4.6.
63. 4.8×4.2.
64. 5.1×5.9.
65. 6.2×6.8.
66. 7.3×7.7.
67. 10.5×10.5.
68. 11.5×11.5.
69. 9.9×9.1.

Deduce the rule for the multiplication of the following numbers:

70. 45×45.
71. 55×55.
72. 65×65.
73. 41×49.
74. 52×58.
75. 63×67.
76. 34×36.
77. 27×23.
78. 82×88.
79. 75×75.
80. 98×92.
81. 37×33.

To divide by $3\frac{1}{3}$ or $2\frac{1}{2}$ point off one decimal and multiply by 3 or 4, respectively; to divide by $12\frac{1}{2}$, $16\frac{2}{3}$, 25, $33\frac{1}{3}$, point off two decimals and multiply by 8, 6, 4, or 3, respectively.

Give quotients:

1. $30 \div 3\frac{1}{3}$.
2. $40 \div 2\frac{1}{2}$.
3. $91 \div 3\frac{1}{3}$.
4. $22 \div 2\frac{1}{2}$.
5. $42 \div 3\frac{1}{3}$.
6. $800 \div 12\frac{1}{2}$.
7. $700 \div 16\frac{2}{3}$.
8. $900 \div 25$.
9. $600 \div 33\frac{1}{3}$.
10. $211 \div 25$.
11. $8 \div .125$.
12. $7 \div .125$.
13. $9 \div .125$.
14. $6 \div .125$.
15. $2 \div .125$.
16. $11 \div 2.5$.
17. $21 \div 2.5$.
18. $31 \div 2.5$.
19. $41 \div 2.5$.
20. $51 \div 2.5$.

Give answers: $(8 \times 83 \times 12\frac{1}{2} \equiv 100$ times $83)$.

21. $8 \times 83 \times 12\frac{1}{2}$.
22. $3 \times 74 \times 33\frac{1}{3}$.
23. $6 \times 59 \times 16\frac{2}{3}$.
24. $4 \times 27 \times 25$.
25. $8 \times 63\frac{1}{2} \times 1\frac{1}{4}$.
26. $6 \times 34\frac{1}{5} \times 1\frac{2}{3}$.
27. $9 \times 17\frac{1}{2} \times 1\frac{1}{9}$.
28. $4 \times 44\frac{1}{5} \times 2\frac{1}{2}$.
29. $37\frac{1}{2}+84\frac{3}{8}+62\frac{1}{2}$.
30. $33\frac{1}{3}+59\frac{3}{4}+66\frac{2}{3}$.
31. $25+17\frac{4}{5}+75$.
32. $87\frac{1}{2}+24\frac{7}{8}+12\frac{1}{2}$.
33. $49\frac{3}{4}-18\frac{7}{8}+28\frac{7}{8}$.
34. $67\frac{5}{8}+25\frac{2}{3}-35\frac{2}{3}$.
35. $33\frac{1}{3}-16\frac{4}{5}+46\frac{4}{5}$.
36. $56\frac{5}{6}+33\frac{3}{4}-53\frac{3}{4}$.
37. $454-75-25$.
38. $576-66\frac{2}{3}-33\frac{1}{3}$.
39. $637-12\frac{1}{2}-87\frac{1}{2}$.
40. $709-62\frac{1}{2}-37\frac{1}{2}$.
41. $6325 \div 4 \div 25$.
42. $8950 \div 3 \div 33\frac{1}{3}$.
43. $4375 \div 8 \div 12\frac{1}{2}$.
44. $5783 \div 6 \div 16\frac{2}{3}$.

Formulas

A formula is a statement of a rule abbreviated by the use of letters and signs.

Thus the area of a rectangle may be expressed by the following formula, in which the area is represented by A, the length by l and the width by w:

$$(1)\quad A = l \times w.$$

Representing the parallel sides of a trapezoid by l and l', respectively, and d representing the perpendicular distance between them, the method of obtaining its area may be indicated thus:

$$(2)\quad A = \frac{l + l'}{2} \times d.$$

The fact that the interest on a certain principal is obtained by multiplying the latter by the rate expressed as hundredths by the time in years is expressed by the following formula:

$$(3)\quad I = P \times \frac{R}{100} \times T.$$

The area of a trapezium is indicated by the following formula, in which d represents the diagonal and p and p' the perpendiculars let fall on it from the vertices of the opposite angles:

$$(4)\quad A = d \times \frac{p + p'}{2}.$$

The convex surface (C. S.) of a right regular prism may be indicated as follows, the number of sides in each base being represented by n, the length of each side by l, and the height of the prism by h:

$$(5)\quad \text{C. S.} = h \times nl.$$

In the case of a square prism, the foregoing becomes

$$\text{C. S.} = 4\,hl.$$

and the entire surface is given in the following:

$$\text{E. S.} = 4\,hl + 2\,l^2.$$

SIGHT PROBLEMS

The arithmetical value represented by any letter in the foregoing formulas may be obtained by replacing the remaining letters by arithmetical numbers and solving the resulting equations.

1. Give the area of a trapezoid when $l = 20$ feet, $l' = 30$ feet, and $d = 10$ feet.

2. When $A = 3\frac{1}{7} R^2$, what is the area of a circle whose radius is 7 yards?

3. If $P = \$300$, $R = 5\,\%$, $T = 2\frac{1}{3}$ years, give the value of I.

4. What is the area of a trapezium whose diagonal measures 20 rods, and having perpendiculars from the vertices to the diagonal measuring 15 and 25 rods, respectively?

5. Give the distance between the parallel sides of a trapezoid when its area is 300 square yards and the parallel sides measure 10 and 20 yards, respectively.

6. What is the formula for the perimeter of a rectangle, whose dimensions are a and b, respectively?

7. State the formula for the volume of a cylinder when the radius of the base is r and its altitude is h.

8. What is the height of a hexagonal prism whose convex surface is 84 sq. ft., each side of the base measuring 2 ft.?

9. The entire surface of a square prism is 360 sq. in.; one side of the base is 10 in. (*a*) What is the convex surface? (*b*) What is the height of the prism?

10. (*a*) Give the formula for the entire surface of a cube whose edge is e. (*b*) What is the entire surface of a cube whose edge is 10 inches? (*c*) What is the edge of a cube whose entire surface is 24 sq. ft.?

11. If the area of a circle is $12\frac{4}{7}$ sq. yd. (*a*) what is the radius? (*b*) The diameter?

WRITTEN PROBLEMS

1. The perpendicular distance between the parallel sides of a trapezoid is 18 feet, one parallel side is 25 feet, and the area is 378 square feet. Find the other parallel side.

METHOD

Substituting 378 for A, 25 for l, and 18 for d in formula (2), the following equation is obtained:

$$378 = \frac{25 + l'}{2} \times 18.$$

Simplifying the second term, and transposing:

$$\frac{450 + 18\,l'}{2} = 378.$$

Clearing of fractions, $450 + 18\,l' = 756.$

Transposing, $18\,l' = 756 - 450 = 306.$

Whence $l' = 17.$

17 ft. *Ans.*

Test: $\frac{17\text{ (ft.)} + 25\text{ (ft.)}}{2} \times 18\text{ (ft.)} \equiv \frac{42\text{ (ft.)}}{2} \times 18\text{ (ft.)}$

$\equiv 21\text{ (ft.)} \times 18\text{ (ft.)} \equiv 378\text{ (sq. ft.)}.$

2. In what time will \$400 yield \$31½ interest, at 3½%?

METHOD

Substituting 31½ for I, 400 for P, and 3½ for R in formula (3), the following equation is obtained:

$$31\tfrac{1}{2} = 400 \times \frac{3\frac{1}{2}}{100} \times T.$$

Simplifying the second member, and transposing,

$$14\,T = \tfrac{63}{2}.$$

Clearing of fractions, $28\,T = 63.$

Whence $T = 2\tfrac{1}{4}.$

2 yr. 3 mo. *Ans.*

Test: $\$400 \times .035 \times 2\tfrac{1}{4} \equiv \$31.50.$

3. One diagonal of a quadrilateral is 46 feet long and the perpendiculars to it from the opposite angles are 24 feet and 32 feet, respectively. Find the area.

4. At what rate will $180 yield $22.50 interest in 2 years 6 months?

5. Find the length of the second perpendicular let fall upon the diagonal of a trapezium whose area is 1134 square rods, when the other perpendicular measures 19 rods and the diagonal measures 42 rods.

6. What principal will produce $45.50 interest at $3\frac{1}{2}\%$ in 3 years 3 months?

7. (*a*) Give the formula for the convex surface of a rectangular prism (representing it by C. S.), when the sides of the base are b and b', respectively, and the altitude is a. (*b*) Give the formula for E. S., the entire surface.

8. The entire surface of a square prism is 312 square inches and one side of the base is 6 inches. What is the altitude?

9. Express the volume (*a*) of a cylinder, the radius of the base being r and its altitude $2r$. (*b*) Of a cone having the same dimensions.

10. The surface of a sphere is equal to the product of the circumference by the diameter. Its volume is equal to the product of the surface by one third of the radius. Representing the diameter by d, find (*a*) its surface, (*b*) its volume. Representing the radius by r, find (*c*) its surface, (*d*) its volume.

11. What per cent of 935 is 374?

SOLUTION

Let $x = \text{rate.}$

Then, $\frac{x}{100} \times 935 = 374,$

or, $\frac{935x}{100} = 374.$

Clearing of fractions, $935x = 37{,}400.$

Dividing, $x = 40.$ *Ans.* 40%.

Test.

12. What number increased by 16 % of itself amounts to 986?

SUGGESTION. $$x + \frac{16x}{100} = 986.$$

13. $\frac{2}{3}$ is what per cent of $\frac{3}{4}$?

$$\frac{2}{3} = \left[\frac{x}{100} \times \frac{3}{4} \equiv\right] \frac{3x}{400}.$$

NOTE. Perform indicated multiplication before clearing of fractions.

14. What per cent profit is made on an article bought for $33\frac{1}{3}$¢ and sold for $37\frac{1}{2}$¢?

$$33\tfrac{1}{3} + \left(\frac{x}{100} \times 33\tfrac{1}{3}\right) = 37\tfrac{1}{2},$$

that is, $$33\tfrac{1}{3} + \frac{x}{3} = 37\tfrac{1}{2},$$

or $$\frac{100}{3} + \frac{x}{3} = \frac{75}{2}.$$

15. What was the cost of an article on which 3 % was lost by selling it for $33.95?

$$x - \frac{3x}{100} = 33.95.$$

Clearing of fractions, $100x - 3x = 3395$, *Etc.*

16. A merchant gained 5 % on a sale of carpet amounting to $ 82.32. What did it cost?

17. Selling price $ 44.70; gain $6\frac{3}{7}$ %. Cost?

18. Cost $ 134.40; loss $ 16.80. Selling price? Loss %?

19. How should goods that cost $ 84 be marked so that they may be sold 20 % below the marked price and still yield a profit of 20 %?

SOLUTION

Let x = marked price.

Then $\frac{4x}{5}$ = selling price.

$$\frac{4x}{5} = 84 + \frac{84}{5}.$$

20. Goods costing $1.60 per yard were sold at a reduction of 25 % from the marked price, and yielded a profit of $12\frac{1}{2}$ %. (*a*) What price were they marked? (*b*) What per cent above cost?

21. At what per cent above cost must goods be marked in order that they may give a profit of $12\frac{1}{2}$ % when they are sold at 25 % below the marked price? What would be the cost of goods marked $2.10 per yard?

22. The interest on $400 for 63 days at x per cent is $3.15. Find the value of x.

$$\frac{400 \cdot 63 \cdot x}{360 \cdot 100} = \frac{315}{100}.$$

Reject 100 in the denominator of each member, simplify first member, etc.

23. At what rate will $360 yield $9.35 interest in 6 months 7 days?

In the formula $P \times \frac{R}{100} \times T = I$, substitute for P, 360; for R, x; for T, $\frac{187}{360}$; for I, $\frac{935}{100}$; as follows:

$$360 \times \frac{x}{100} \times \frac{187}{360} = \frac{935}{100}.$$

Rejecting 100 in both members, canceling, etc., $187x = 935$.

In the following seven examples find the value of x:

	PRINCIPAL	RATE	TIME	INTEREST
24.	$400	x	327 da.	$16.35
25.	$500	6 %	x	$39.75
26.	x	$4\frac{1}{2}$ %	84 da.	$37.80
27.	$300	x	693 da.	$57.75
28.	$700	6 %	x	$90.65
29.	x	5 %	285 da.	$85.50
30.	$800	x	918 da.	$86.70

31. A piano whose "list" price was \$ 600 was sold at a discount of 25 % and an additional discount of x % on the sum remaining after the deduction of the first discount. What is the value of x if the net price obtained for the piano was \$ 427.50?

The first discount of 25 % leaves a remainder that is 75 % of \$ 600, or \$ 450. The second discount of x % leaves a remainder which may be indicated as $450 - \frac{x}{100}$ of 450.

$$450 - \frac{450x}{100} = \frac{42750}{100}. \qquad \textit{Etc.}$$

32. The successive discounts on a piano listed at \$ 600 are x % and 25 %. Find the value of x, when the net price is \$ 427.50.

$$\left(600 - \frac{600x}{100}\right)\frac{3}{4} = 427\tfrac{1}{2}.$$

33. The net price of an article after the deduction of successive discounts of 33⅓ and 10 % is \$ 270. What is the list price?

$$x(1 - \tfrac{1}{3})(1 - \tfrac{1}{10}) = 270.$$

In the following eleven examples find the value of x:

	List Price	Successive Discounts	Net Price
34.	\$ 400	x and 25 %	\$ 180
35.	x	20 and 20 %	\$ 640
36.	\$ 500	33⅓ and x %	\$ 300
37.	x	40 and 20 %	\$ 960
38.	\$ 600	x and 5 %	\$ 456
39.	x	30 and 10 %	\$ 504
40.	\$ 700	50 and x %	\$ 329
41.	x	25 and 25 %	\$ 144
42.	\$ 800	x and 10 %	\$ 648
43.	x	33⅓ and 10 %	\$ 450
44.	\$ 900	20 and x %	\$ 576

MISCELLANEOUS SIGHT PROBLEMS

1. The sum of two numbers is 72; one is 5 times the other. Find the larger number.

2. What is the smaller of two numbers, it being one sixth of the larger, and the sum of the numbers being 70?

3. Find the product of two numbers whose sum is 70 and whose quotient is 6.

4. The difference of two numbers whose ratio is 1 to 6 is 50. What is their sum?

5. If two persons are 40 miles apart, how many hours will it take them to meet if one travels at the rate of 4 miles per hour, and the other at the rate of 6 miles per hour, both starting at the same time and traveling toward each other?

6. The sum of two numbers is 40; one is $1\frac{2}{3}$ times the other. What is the smaller number?

7. Find the smaller of two numbers whose sum is 40, one being $\frac{3}{5}$ of the other.

8. What is the product of two numbers if their sum is 40, and the numbers bear the ratio of 3 : 5?

9. The product of two numbers, whose ratio is 3 : 5, is 60. Find the larger number.

10. What is the square of $x - 3$?

11. Extract the square root of $9 - 6x + x^2$.

12. Find the value of x in each of the equations:

$$(a)\ \frac{x}{3} + 17 = 48, \quad (b)\ x = 31 + \frac{2x}{3}.$$

13. If $\frac{2}{7}$ of a number be added to the number itself, the result is 99. What is $\frac{2}{7}$ of the number? What is $\frac{6}{7}$ of the number?

14. Divide \$140 between two people giving one \$60 more than the other. What is the share of each?

15. Find a number whose half exceeds its third by 20.

16. A man paid \$360 for a horse and a wagon, giving one half as much for the former as for the latter. What did he pay for each?

17. A can mow a field in 30 days, and B can mow it in 20 days. How long would it take both together?

$$\frac{1}{20}+\frac{1}{30}=\frac{1}{x}.$$

18. A and B can do a piece of work in 12 hours, which it would require A 30 hours to do, working alone. How many hours would B require, working alone?

19. A man paid \$30 for an article, which was $\frac{3}{4}$ of $\frac{4}{5}$ of the marked price. What was the marked price?

20. Two boys divided a number of marbles, one receiving 2 more than one half the number, and the other receiving 2 more than one fourth of the number. How many marbles did each get?

$$x=\frac{x}{4}+2+\frac{x}{2}+2.$$

21. Solve the following equation:

$$12x-3x-2x=63+7.$$

22. Given $4x^2-6=2x^2+66$, to find the value of x.

23. My money increased by $\frac{12}{100}$ of my money is \$112. How much money have I?

24. A person's money increased 6% of itself each year for two years amounts at the end of that time to \$112. How much did he have originally?

25. A number increased by $\frac{3}{7}$ of itself and then decreased by 12 becomes 18. What is the number?

26. M and N together have \$140, M having \$10 more than N. How many dollars has each?

27. One pipe can fill a tank in 2 hours, and another can fill it in 3 hours. In how many hours and minutes can both together fill it?

28. Solve: $\frac{x}{2}+\frac{x}{3}+6=x$.

29. One half a pole is in the water, one third is in the mud, and 6 feet extend above the water. How long is the pole?

30. The perimeter of a rectangular field is 60 rods and one side is 10 rods longer than the adjacent side. How many square rods in the field?

31. Find the roots of the equation: $x^2+10x=24$.

32. What is the difference in latitude between a place 30° north of the equator and one 40° south of the equator?

33. Two men 80 miles apart are traveling towards each other, one going $4\frac{1}{2}$ miles per hour and the other $5\frac{1}{2}$ miles per hour. In how many hours will they meet? How far will each travel?

34. If $x+\frac{x}{5}=60$, what is the value of x?

35. A man sells an article at 20% more than the cost, receiving $60 for it. Find the cost.

36. The cost of a certain quantity of wheat added to a commission of $\frac{1}{50}$ of the cost, amounts to $102. (*a*) What was the cost of the wheat? (*b*) What was the commission?

37. If a miller as a charge for grinding deducts $\frac{1}{8}$ of the corn sent him to be ground, and he returns 490 pounds to the owner of the corn, how many pounds did the latter send?

38. A person pays $16 for a coat and a vest, the cost of the latter being $\frac{1}{3}$ that of the former. What was the price of each?

39. N has paid $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of a debt and still owes $20. What was the original debt?

40. Solve: $x-\left(\frac{x}{3}+\frac{x}{4}+\frac{x}{6}\right)=30$.

41. One half of 20 is two thirds of what?

42. The product of two equal factors is 100. What are the factors?

43. One number is three times another and their product is 300. What are the numbers?

44. Solve: $8x + 3 = 9x - 7$.

45. When a farmer put 8 sheep in each pen, he had 3 sheep over; to put 9 in each pen he would require 7 more sheep. (*a*) How many pens had he? (*b*) How many sheep?

46. If 4 marbles are given to each of a number of boys, there are 2 marbles remaining, while 2 more marbles would be needed to make each boy's share 5 marbles. How many boys are there?

47. Twenty per cent of 20 per cent of a number is 4. What is the number?

48. Find the value of x, when

$$\tfrac{1}{5} \text{ of } \tfrac{1}{5} \text{ of } x = 1.$$

49. Find the value of x, when

$$(\tfrac{1}{5}x)(\tfrac{1}{5}x) = 16.$$

50. The square of one fifth of a number is 36. What is the number?

51. Twenty per cent of a number multiplied by 30 per cent of the same number is 6. Find the number.

52. Find the value of x, when

$$\frac{x}{5} \times \frac{3x}{10} = 24.$$

53. Two men, 100 miles apart, are traveling towards each other at the hourly rate of $4\frac{1}{2}$ miles and $5\frac{1}{2}$ miles, respectively. In how many hours will they be 20 miles apart before they meet? In how many hours will they be 20 miles apart after they pass each other?

54. What number must be added to 30 and to 6, respectively, in order that the latter sum may be 4 times the other?

MISCELLANEOUS WRITTEN PROBLEMS

1. The sum of two numbers is 85. One of the numbers is $3\frac{1}{4}$ times the other. Find the product of the numbers.

Before commencing to *state* a problem in terms of x, the pupil must determine for what the x is to stand. He must not assume, for instance, that x is to represent in the foregoing problem the product of the numbers. He should use x to represent the smaller number. After the numbers are found, the product is readily calculated.

2. The sum of two numbers is 144, one of them being $3\frac{1}{2}$ times the other. What is the difference between them?

3. The difference between two numbers, one of which is one sixth of the other, is 125. Find the sum of the numbers.

4. A, B, and C have together 85 sheep. How many has each, if A has 10 more than C and 7 more than B?

If x is employed to represent the number of C's sheep, and $x + 10$ to represent the number of A's, what will represent the number of B's?

5. The difference between $\frac{1}{2}$ a number and $\frac{4}{25}$ of the same number is 51. What is the number?

6. The difference between 50% of a number and 16% of the same number is 51. Find the number.

$$\frac{50x}{100} - \frac{16x}{100} = 51.$$

7. A certain sum was paid for the expenses of a trip, which 6 persons agreed to share. Two having declined to pay, each of the others was compelled to pay $2 more. What were the expenses?

8. Find the value of x in the following:

$$\frac{2x+10}{5} - 7 = 3.$$

9. If 10 be added to twice a number, the sum be divided by 5, and 7 be deducted from the quotient, the remainder is 3. What is the number?

10. Find the value of x:

$$\tfrac{1}{4}(x-20)=\tfrac{1}{3}x-100.$$

11. One fourth of what a dealer had remaining after selling 20 tons of coal is 100 tons less than one third of the amount he had at first. How many tons had he at first?

12. E has 100 acres of land and F has three times as many. How many additional acres must each buy so that E will have one half as much land as F?

13. A has 200 acres of land and B has 500. How many acres must A buy from B so that each may have the same quantity?

14. How many sheep must B, who has 50, sell to A, who has 20, in order that A may have $\frac{2}{3}$ as many as B?

15. What number must be added to 36 and to 12 in order that one may be $2\frac{1}{2}$ times the other?

16. In how many years will a boy, now 12, be two thirds as old as his father who is now 36?

17. In what year will a boy born in 1912 be two thirds as old as his father, who was born in 1888?

18. A is now 24 years older than B, who is 16. In what time will A's age be $1\frac{1}{2}$ times B's age?

19. How is the number 75 represented by x and y, when x represents the tens' figure (7) and y represents 5, the units' figure?

20. How is the number 57 represented when $x=7$ and $y=5$?

21. Solve the following equations:

$$10x+y-18=x+10y.$$
$$x+y=12.$$

22. A certain number consists of two digits. The sum of the digits is 12. When 18 is subtracted from the number, the order of the digits is reversed. What is the number?

23. The difference between the digits composing a number of two figures is 2. When 18 is added to the number, the order of the digits is reversed.

Let $x =$ tens' figure,
$y =$ units' figure,
$y - x = 2$. (Why not $x - y = 2$?)

24. From $8x + 14y - 8$ take $3x - 8y - 7$.

25. Solve $(x + 5)^2 = (4 - x)^2 + 21x$. Verify your result.

26. If 24 be added to a certain number, the sum is 3 times the number. What is the number?

27. Divide 24 into two such parts that the sum of the quotients of one part divided by 7 and the other by 3 will be 4.

28. A has 50 acres of land more than B, and the latter has 60 acres more than C. The three have together 470 acres. How many acres has each?

29. A has 110 acres more than C and 50 more than B, the three having 470 acres. How many acres has each?

30. If 24 more than twice a number be deducted from 80, the remainder is 2 more than the number. Find the number.

31. Find a number such that when it is diminished by 5, two thirds of the remainder will be 15 less than the number.

32. Determine the value of x and y in the simultaneous equations:

$$5x + 6y = 29 \text{ and } 3x + 2y = 11.$$

33. The sum of two numbers is 27, and 4 times the first added to three times the second is 93. Find the numbers by employing x alone, and also by the use of x and y.

34. A number of two digits is equal to 4 times the sum of the digits, and the digits are reversed when the number is increased by 18. What is the number?

35. A pole is 37 feet long. At the top is a flag 6 feet wide. The height of the bottom of the flag above the ground is equal to 6 feet more than the length of the pole under the ground. How far above ground is the top of the flag?

36. A dealer sold 25 crates of strawberries at $3 each, but some of the crates proving unsatisfactory, he charged only half price for the latter. If the corrected bill was $66, how many crates were unsatisfactory?

37. A boy bought some oranges at 3 cents each and 3 times as many pears at 2 cents each. He sold them all at 3 for 10 cents, thereby gaining 52 cents. How many did he buy of each?

38. Two watches that were correct at noon yesterday, now differ one half hour. What is the correct time if one loses 2 minutes in 3 hours and the other gains 20 minutes in 24 hours?

39. A can do a piece of work in 10 days, B in 30 days, and C in 15 days. In how many days can the three do the work? In how many days can A and B do it? B and C? A and C?

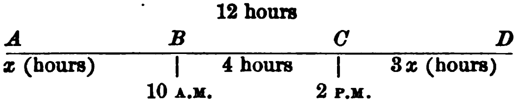
40. M can perform a task in $1\frac{2}{3}$ days, while N would require $2\frac{1}{2}$ days to do it. How long would both, working together, require?

$$\frac{1}{1\frac{2}{3}}+\frac{1}{2\frac{1}{2}}=\frac{1}{x}.$$

41. Divide 44 into two parts such that one part increased by 3 will be one half as great as the remainder left after the other part is diminished by 5.

42. The owner of a gold and a silver watch bought a chain for $10. The value of the gold watch and the chain is 4 times the value of the silver watch. The value of the silver watch and the chain is one half the value of the gold watch. What is the value of each watch?

43. At 10 A.M. a train has already taken one third of the time it will require after 2 P.M. for its regular 12-hour trip. At what time did it start?



44. Two hours ago the number of hours after noon was 3 times as great as will be the number of hours to midnight two hours from now. What time is it?

45. A has 5 cows more than B. If A should buy four cows from B, the former would then have twice as many as the latter? How many has each?

46. One girl is 6 years older than another. Six years ago the younger was one fourth as old as the other will be 6 years from now. Find the age of each.

47 A, B, and C can do a piece of work in 10 days, which B and C can do in 15 days. How many days would A require to do it alone?

$$\frac{1}{10}-\frac{1}{15}=\frac{1}{x}.$$

48. A traveler on a 6 days' trip found that $\frac{3}{5}$ of the time that had elapsed before midnight of July 3 was equal to $\frac{3}{7}$ of the time remaining. How many hours had elapsed? When did he begin his trip?

49. The captain of a vessel found at noon of July 4 that the trip had taken as many hours as would be required after midnight of July 6 to complete the trip in 144 hours. When did the vessel start?

50. At noon July 4, the captain of a vessel found that 12 hours previously he had spent as much time as would be left 12 hours after midnight of July 6 to complete the trip in the schedule time of 6 days. When was the vessel due at her destination?

PART TWO

CHAPTER I

SYMBOLS AND OPERATIONS

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Symbols and Expressions

Algebra, like arithmetic, deals with numbers; it differs, however, from arithmetic in employing letters, as well as figures, to represent numbers, and in employing negative numbers as well as positive ones.

An expression consisting of letters or letters and figures is called an *algebraic number;* one consisting exclusively of figures is called an *arithmetical number.*

Roman numbers are not algebraic, since the letters composing them are, in reality, figures.

POSITIVE AND NEGATIVE NUMBERS

Positive numbers are denoted by a prefixed + (plus) sign or by the absence of a sign; negative numbers are denoted by a prefixed − (minus) sign.

PRELIMINARY EXERCISES

To distinguish between + and − used as marks of quality from the same signs used to indicate operations, the former are sometimes written near the top of the number or letter. $(^+6)-(^-3)$ may be read "positive 6 minus negative 3."

1. If distance north is indicated by $^+$, what mark may be employed to indicate distance south?

2. A man drives 10 miles due north, then 4 miles due south. (*a*) How many miles has he gone? (*b*) $10+4\equiv$? (*c*) How far is he from his starting point? (*d*) $(^+10)+(^-4)\equiv$? (*e*) Is he north or south of the starting point?

3. A man drives 10 miles north, 4 miles south, and 8 miles south. (*a*) How many miles has he driven? (*b*) $10+4+8\equiv$? (*c*) How far is he from his starting point? $(^+10)+(^-4)+(^-8)=$? Is he north or south of the starting point?

4. Find the difference in latitude between two places in 70° and 40° north latitude, respectively. $(^+70)-(^+40)\equiv$? Find the difference between a place in latitude 70° north and one in latitude 40° south. $(^+70)-(^-40)\equiv$?

5. Express without signs of quality the subtraction of $^+40$ from $^+70$; the subtraction of $^-40$ from $^+70$.

6. A has $9 and B has $6. How many dollars have both? How much more has A than B? How much less has B than A?

7. M has $9 and N owes $6. How much better off is M than N?

SIGHT EXERCISES

Give answers:

1. $(^+9)+(^+6)$.
2. $(^+9)-(^+6)$.
3. $(^+9)+(^-6)$.
4. $(^-9)+(^-6)$.
5. $(^+9)-(^-6)$.
6. $(^-9)-(^-6)$.
7. $(^-9)-(^+6)$.
8. $(^-9)+(^+6)$.
9. $(^+6)+(^+9)$.
10. $(^+6)+(^-9)$.
11. $(^+6)-(^+9)$.
12. $(^+6)-(^-9)$.
13. $(^-6)+(^-9)$.
14. $(^-6)-(^-9)$.
15. $(^-6)+(^+9)$.

ALGEBRAIC EXPRESSIONS

A number represented either by letters, or by letters and arithmetical numbers, or by a group of algebraic symbols, is called an **algebraic expression.**

(I) $3a^3$, (II) $-4xy$, (III) $-a(m+n)$, (IV) $\frac{ab-d}{c}$, and (V) $3a^3-4xy+\frac{ab-d}{c}-a(m+n)$ are algebraic expressions, each of the first four being a **simple** expression, and the fifth being a **compound** expression.

A simple algebraic expression is called a **term**; a compound algebraic expression contains two or more terms connected by one or more plus or minus signs, or by both.

The compound expression (V) is composed of four terms; as $\frac{ab-d}{c}$ and $a(m+n)$ is each considered as a single term, the former being an indicated division and the latter an indicated multiplication. Terms of this kind are called **compound terms.**

An algebraic expression of one term is called a **monomial**; of more than one term, a **polynomial.** Polynomials of two, of three, and of four terms are called **binomials, trinomials,** and **quadrinomials,** respectively.

$3a^3$ is a monomial, $3a^3-4xy$ is a binomial, $3a^3-4xy-a(m+n)$ is a trinomial. The latter becomes a quadrinomial when it is written $3a^3-4xy-am-an$.

A term is generally understood to include its prefixed sign, being positive when the sign is + or when there is no prefixed sign, and negative when the sign is −. In the quadrinomial given above (V), $3a^3$ and $+\frac{ab-d}{c}$ are positive, while $-4xy$ and $-a(m+n)$ are negative, the character of a fraction or of a parenthesis being determined by the sign prefixed to the whole term.

ALGEBRAIC SYMBOLS

Addition and subtraction are indicated in algebraic operations by the use of the + and the − signs, respectively.

The binomial $a-2b$ means a diminished by $2b$; it may also denote, however, the algebraic sum of a and $-2b$.

A dot ($\cdot$) between two numbers indicates that they are to be multiplied, but the product of literal quantities is denoted by writing the letters without an intervening sign.

In writing the product of $b \cdot 3 \cdot a \cdot 4$, the numerical quantities are combined and written first, the letters being generally written in alphabetical order, $12\,ab$.

No sign is required to indicate that an expression within a parenthesis or other sign of aggregation is to be multiplied by a term placed before or after it.

Thus, $3 \times a \times b$, or $3 \cdot a \cdot b$, means 3 multiplied by a multiplied by b, the result being written $3\,ab$; $4\,(m + 6\,n)$ means 4 times the sum of m and 6 times n, which is equal to $4\,m + 24\,n$; $(a + 2\,b)(c + 3\,d)$ means that the sum of a and $2\,b$ is to be multiplied by the sum of c and $3\,d$, the absence of a sign between the parentheses indicating that their product is to be taken.

When the same factor occurs more than once in an algebraic term, an **exponent** is written above it to the right, to indicate the number of times it is so employed.

Thus, $3\,a^3b^2x \equiv 3 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot x$;

$$4(m + 2\,n)^3 \equiv 4\,(m + 2\,n)(m + 2\,n)(m + 2\,n).$$

An exponent is also called an **index**.

The division of one quantity by another is indicated by writing the divisor as the denominator of a fraction, of which the dividend is the numerator.

Thus, the quotient of a divided by b, or $a \div b$, is written $\frac{a}{b}$; $\frac{a + 2\,b}{3\,c - d}$ means that the sum of a and $2\,b$ is to be divided by the difference between $3\,c$ and d.

The parenthesis ( ), the bracket [ ], the brace { }, are used as signs of aggregation, the different forms being employed to prevent confusion when signs of aggregation are required within other such signs.

A line drawn above (or below) two or more terms to be united, is sometimes used as a sign of aggregation. A vertical line is similarly employed.

A sign $\surd$ indicates the root of the number before which it is written, an **index** placed at the opening indicating the number of the root. When no index is given, the square root is meant.

Thus: $\sqrt{x}$ indicates the square root of x; $\sqrt[3]{a^2+b}$, the cube root of the sum of a^2 and b; $\sqrt[4]{x-y}$, the fourth root of the difference of x and y.

While any factor of a term may be considered a **coefficient**, the word is frequently limited to the numerical coefficient, if there be one.

Thus: in the term $4\,a^3b^2x$, 4 is the coefficient of a^3b^2x; $4\,a^3$, of b^2x; $4\,a^3b^2$, of x; $4\,b^2$, of a^3x; etc. In the expression, $3(a+bx)^3$, 3 is the coefficient of the quantity in the parenthesis.

The operation of multiplication or of division is also indicated by the sign used in arithmetic, viz. $\times$ or $\div$, respectively; but the dot ($\cdot$) is preferred, owing to the resemblance of the $\times$ to the letter x.

In this country the dot is written above the line to distinguish it from the decimal point; in countries that use the comma for a decimal point, the dot is placed on the line.

To denote the identity of two expressions, the sign $\equiv$ may be placed between them, the sign of equality, $=$, being used to connect two members of an equation.

Another form of the sign of equality is the double colon (: :) used in this country to indicate the equality of two ratios.

The sign of inequality, $>$ or $<$, denotes that two expressions are unequal. Thus, $a > b$ is read, "a is greater than b"; $b < a$ is read, "b is less than a," the quantity placed at the open part of the sign being the greater.

The sign $\therefore$ means **hence, or therefore.**

Thus,
$$\begin{aligned} 3\,a &= 15;\\ \therefore\quad a &= 5.\end{aligned}$$
is read, "3 a is equal to 15; hence, a is equal to 5."

The sign $\because$ means **since, or as.**

Thus,
$$\begin{aligned} \because\ 3\,a &= 15;\\ \therefore\quad a &= 5.\end{aligned}$$
is read, "since 3 a is equal to 15, therefore a is equal to 5."

SIGHT PROBLEMS

1. At a dollars apiece what will 3 coats cost? b coats? $2b$ coats? How many can be bought for $2a$ dollars? For ab dollars? For $5ab$ dollars?

2. A man bought 3 coats at c dollars each and 2 coats at a dollars each. How much did he pay for all? Give the total cost of a coats at c dollars each and $2b$ coats at the same rate.

3. A piece of cloth a yards long is bought at b dollars per yard. What is the profit per yard if it is sold at c dollars per yard? What is the profit on the whole piece?

4. A mixture is made of a pounds at m cents per pound, and b pounds at n cents per pound. What is the total cost of the mixture? How many pounds are in the mixture? What is the cost of the mixture per pound?

5. In the number 72, what does the 7 represent? The 2? How can 72 be represented by x and y, when $x = 7$ and $y = 2$? With the same values of x and y, how is the inverted number (27) represented? What is the sum of the digits of 72? Of 27? Represent the sum of the digits by x and y, these letters having the respective values given above.

6. If a man's present age is represented by x, what was his age 10 years ago? What will be his age 10 years hence?

7. Two trains start from the same place at the same time going in the same direction, the first going a miles per hour, the second going b miles per hour. How far does the first go in c hours? How far does the second go in c hours? How far apart are they in c hours?

8. Two trains start at the same time from points a miles apart, and go away from each other, one at the rate of b miles per hour and the other at the rate of c miles per hour. How far apart are they in an hour? How far does the first go in d hours? How far does the second go in d hours? How far apart are they in d hours?

SIGHT EXERCISES

Give the values of the following expressions, when

$$a=1, \quad b=2, \quad c=3, \quad m=4, \quad n=5.$$

1. $4ab$.
2. cmn.
3. $c+mn$.
4. $\frac{2n-4}{c}$.
5. $(ab)^2$.
6. a^2b^2.
7. a^2+b^2.
8. b^2-ab.
9. $(b+c)^2$.
10. $\frac{a+2b}{n}$.
11. $2c(m-n)$.
12. $a+(2b)^2$.
13. $4ab-n$.
14. $3abc$.
15. $(a+2b)^2$.
16. $\frac{n\sqrt{m}}{ab}$.
17. $a+2b^2$.
18. $(2b-a)^2$.

WRITTEN EXERCISES

Find the value of each:

1. $a(b+2c)-m(n^2-1)+2a^2$.
2. $(a+2b^2)(m-n)+2n\sqrt{m}+1$.
3. $[(a+2b)^2+2c(m-n)](n-2b+a)$.
4. $(b+a)(b-a)(b^2-a^2)-3ab$.
5. $[(m+n)-(c+b)+a]-(\sqrt{a+c}-b)$.
6. $\frac{a+b}{c}+\frac{2m-n}{b}+\frac{6c}{m}-\sqrt{2bm}$.
7. $\frac{2c+m}{b+c}-(a+b)\frac{n-c}{n-b}+\sqrt{4am}$.
8. $\frac{2a+b}{m}+\frac{2n-4}{c}-\frac{n\sqrt{m}}{ab+3}+b^2$.
9. $\frac{1}{b^2}+\frac{am}{2a+b}+\frac{c}{2n-4}-\frac{ab+3}{n\sqrt{m}}$.
10. $(ab+b^2)\frac{2ac+m}{n}-\left(\frac{a+b}{c}\right)^2$.
11. $(a+b+c)^2-\sqrt{\frac{m+n}{b^2}}$.

Transformations

Terms are said to be **like**, or **similar**, when they are represented by the same letters affected by the same exponents, even if they have different signs or different numerical coefficients.

Thus, $4a^2b$ and $-3a^2b$ are similar terms; as are $3bc^2$ and $7bc^2$, $2xyz$ and $-5xyz$. Compound terms such as $4(a+b)$ and $-2(a+b)$, $-5\sqrt{n^2+1}$ and $3\sqrt{n^2+1}$, $2(a+c)^2$ and $7(a+c)^2$, are also like, or similar.

A change in the form of an expression with no change in its value, is called a **transformation.**

The change of x into $\sqrt{x^2}$, of $2a+2a+b$ into $4a+b$, of $\frac{a}{2}$ into $\frac{2a}{4}$, etc., is, in each case, a transformation.

An expression is **simplified** when it is transformed into one containing the smallest possible number of simple terms, each of which is composed of the fewest possible letters.

Terms connected by signs of operation are simplified by performing the indicated operations as far as possible.

Thus, $a+2a$ is simplified into $3a$; $(a+b)(a-b)$, into a^2-b^2; $\frac{6a^2}{2a}$, into $3a$; $\frac{2a}{6a^2}$, into $\frac{1}{3a}$; etc.

An expression consisting of like terms affected by $+$ and $-$ signs is simplified by finding the difference between the total of the positive coefficients and that of the negative ones, writing this as the coefficient of the common portion and prefixing the sign of the greater total.

$$\begin{aligned}\text{Thus, } 6a-2a+3a-4a+5a-9a &\equiv +14a-15a\\ &\equiv -a. \quad \textit{Ans.}\\ 3\sqrt{x+y}+2\sqrt{x+y}-\sqrt{x+y} &\equiv 5\sqrt{x+y}-\sqrt{x+y}\\ &\equiv 4\sqrt{x+y}. \quad \textit{Ans.}\\ -6(a+2b)-4(a+2b)+5(a+2b) &\equiv -10(a+2b)+5(a+2b)\\ &\equiv -5(a+2b). \quad \textit{Ans.}\end{aligned}$$

The simplification of expressions of this kind consists in finding the algebraic sum of the terms.

SIGHT EXERCISES

Simplify the following:

1. $16 - 9 - 5.$
2. $6\,ab^2 - 3\,ab^2.$
3. $6 + 5 - 4 - 3.$
4. $6 - 5 + 4 - 3.$
5. $6\,x + 4\,x - 3\,x.$
6. $3(a+b) - 2(a+b).$
7. $14(a+x) + 7(a+x).$
8. $9\sqrt{x} - 3\sqrt{x} + 2\sqrt{x}.$
9. $15\sqrt{x^2+y} - 12\sqrt{x^2+y}.$
10. $13x\sqrt{y} - 8x\sqrt{y} + 5x\sqrt{y}.$

Algebraic Operations

ALGEBRAIC ADDITION

Addition in algebra consists in combining two or more algebraic expressions into a single one. Similar terms, are united into one term; dissimilar terms, while written separately, are connected by the proper sign into one expression.

The sum of $3\,a$ and $-5\,a$ is $-2\,a$; the sum of $3\,a^2$ and $-5\,a^3$ can only be indicated as $3\,a^2 - 5\,a^3$, the terms being dissimilar.

Like algebraic terms are added by prefixing to the common literal part the algebraic sum of their coefficients.

SIGHT EXERCISES

Give the algebraic sum of each column:

1. $$\begin{array}{r} 6\,ax \\ -5\,ax \\ ax \\ \hline \end{array}$$

2. $$\begin{array}{r} 4(a+b) \\ (a+b) \\ -10(a+b) \\ \hline \end{array}$$

3. $$\begin{array}{r} \sqrt{m} \\ 4\sqrt{m} \\ -2\sqrt{m} \\ \hline \end{array}$$

4. $$\begin{array}{r} -6(a+b)^2 \\ (a+b)^2 \\ -5(a+b)^2 \\ \hline \end{array}$$

5. $$\begin{array}{r} 5\sqrt{n^2+1} \\ -7\sqrt{n^2+1} \\ \sqrt{n^2+1} \\ \hline \end{array}$$

6. $$\begin{array}{r} 4(m+n) - 5\sqrt{r} \\ -3(m+n) + 6\sqrt{r} \\ -2(m+n) - \sqrt{r} \\ \hline \end{array}$$

7. $$\begin{array}{r} 5\,a^2 - 2\,ab \\ 6\,a^2 + 4\,ab \\ -3\,a^2 + ab \\ \hline \end{array}$$

8. $$\begin{array}{r} -7\,ab^2 + b^3 \\ 2\,ab^2 - 2\,b^3 \\ -5\,ab^2 + 3\,b^3 \\ \hline \end{array}$$

9. $$\begin{array}{r} 4\sqrt{x} + 3\sqrt{xy} \\ -\sqrt{x} + 5\sqrt{xy} \\ -2\sqrt{x} + \sqrt{xy} \\ \hline \end{array}$$

10. $$\begin{array}{r} -7\,a^2 - 3\,abc \\ -5\,a^2 - 5\,abc \\ 6\,a^2 - 7\,abc \\ \hline \end{array}$$

WRITTEN EXERCISES

Find the sums of expressions, as follows:

1. $4a^2 - 5ab + 7$, $3ab - 2a^2 - 6$, $10 - ab - 3a^2$, and $4a^2 - 6 + b$.

PROCESS

$$\begin{array}{rrrrl}
4a^2 & -5ab & +7 & & (6) \\
-2a^2 & +3ab & -6 & & (-5) \\
-3a^2 & -ab & +10 & & (6) \\
4a^2 & & -6 & +b & (-1) \\
\hline
3a^2 & -3ab & +5 & +b & \textit{Ans.}
\end{array}$$

In the second line $-2a^2$ is placed in the first column, $+3ab$ in the second, etc. In the third line the order is $-3a^2 - ab + 10$; etc.

The positive total of the first column is $+8a^2$ and the negative total is $-5a^2$; their difference, $3a^2$, is written as the sum, which is positive, since the greater total is positive; etc.

TEST

A rough test of the correctness of the coefficients and signs in the sum is obtained by combining at sight all the coefficients in the addends and comparing the result with the combination of the coefficients in the answer.

Taking the addends the pupil thinks $4 - 5(-1) + 7(6) + 3(9) - 2(7) - 6(1) + 10(11) - 1(10) - 3(7) + 4(11) - 6(5) + 1(6)$, the numbers in the parentheses representing the successive sums, the final result being 6. The sum of the coefficients in the answer is $3 - 3(0) + 5(5) + 1(6)$, which agrees with the first result.

The test figures may be written in a parenthesis alongside each addend, as shown above.

2. $3x^2 - 5x + 4$, $3x - 2x^2 - 5$, $7 + x^2 - 4x$, and 1.
3. $a^2 - b^3$, $2a^2 + 4b^3$, $5a^2 - b^3$, $7b^3$, and $a^2 - b^3$.
4. $8a^2b + 2ab^2 - 3b^3$, $8b^3 - 7a^2b - 4ab^2$, and $5a^2b$.
5. $5mn^2 - 3m^2n$, $4mn^2 + 7m^2n$, and $-2m^2n + 3mn^2$.
6. $3x^4 - 5x^3y$, $-5x^4 + 8x^3y + y^4$, and $7x^4 + 2x^3y + 1$.
7. $2m + 4an$, $-5an + 6m$, $4 - 3an - 5n$.
8. $ab - 4bc$, $-a^2 - ab$, $3ab - 4bc$, $6a^2$.

ALGEBRAIC SUBTRACTION

Subtraction in algebra consists in writing in a single expression the difference between two expressions.

While the difference in temperature between 12° below zero and 18° above zero is 30°, the algebraic difference gives the negative sign to indicate that $-12°$ is the minuend and $+18°$ the subtrahend, the second number being the one to be taken from the other.

In arithmetic the difference between 15 and 25 is the same as the difference between 25 and 15, the smaller number being necessarily taken as the subtrahend.

Similar terms are united into one term as in addition, while dissimilar terms are connected into one expression by the proper sign.

Thus, the difference between $18a$ and $12a$ may be indicated $18a-12a$, which is simplified into $6a$; that between $18a$ and $-12a$ may be indicated $18a+12a$, which is simplified into $30a$; that between $-18a$ and $12a$ may be indicated $-18a-12a$, which is simplified into $-30a$; that between $-18a$ and $-12a$ may be indicated $-18a+12a$, which is simplified into $-6a$.

From the foregoing it will be seen that an example in algebraic subtraction may be changed to one in addition by changing the sign of the subtrahend. When the latter consists of several terms, the sign of each is changed.

Two like algebraic terms are subtracted by prefixing to the common literal part the algebraic sum of the numerical coefficient of the minuend and that of the subtrahend with its sign changed.

SIGHT EXERCISES

Give answers:

$$\begin{array}{llllll} 1. & 8a+2b & 2. & 6a^2+cd & 3. & -7x^2y-3m^2n \\ & \underline{-(5a+b).} & & \underline{-(-3a^2-4cd).} & & \underline{-(-4x^2y+6m^2n).} \end{array}$$

$$\begin{array}{llllll} 4. & -a^2+ab & 5. & a^2+ab & 6. & -3a^2+4ab \\ & \underline{-(a^2-ab).} & & \underline{-(-a^2-ab)} & & \underline{-(-2a^2-4ab).} \end{array}$$

7. $19-(8+7)$. 8. $(19-8)-7$. 9. $19-(8-7)$.

WRITTEN EXERCISES

Find differences:

1. $3a^2 - 4ab - 6b^2$ and $2b^2 + 4ab - a^2 + 1$.

Process

$$\begin{array}{lr}
3a^2 - 4ab - 6b^2 & (-7) \\
-a^2 + 4ab + 2b^2 + 1 & (6) \\
\underline{+ \quad - \qquad - \qquad -} & \\
4a^2 - 8ab - 8b^2 - 1 & \textit{Ans.}
\end{array}$$

Write the subtrahend under the minuend, placing similar terms in the same column. Underneath each term of the subtrahend write the opposite sign and add the two expressions, using the signs of the subtrahend as changed.

By degrees the pupil may omit the change of sign as unnecessary.

TEST

A rough test similar to that employed in addition may be used. This finds the algebraic sum of the coefficients in the minuend, and deducts therefrom the algebraic sum of those in the subtrahend. The result is then compared with the algebraic sum of the coefficients in the answer.

Sum of coefficients in minuend $3 - 4 - 6 = -7$. (a)
Sum of coefficients in subtrahend $2 + 4 - 1 + 1 = 6$. (b)
Sum of coefficients in answer $4 - 8 - 8 - 1 = -13$.
Difference between (a) and (b) $-7 - 6 = -13$.

2. $5x^2 - 3xy + 4y^2 - (2x^2 + 7xy - 5y^2)$.

3. $a^3 + 3a^2b - 4ab^2 + b^3 - (6a^3 + 2a^2b - 9ab^2 - 5b^3)$.

4. From the sum of $5x^3 - 3x^2y + 3xy^2 - 5y^3$ and $2x^3 + 7x^2y - 5xy^2 - 3y^3$, take the sum of $4x^3 + 10x^2y - 11xy^2 + y^3$ and $6x^3 - 9x^2y - 7xy^2 + 4y^3$.

Given

$$\begin{aligned}
D &= 3a^2 - 5ab - 3b^2, \\
E &= 6a^2 + 3ab - 2b^2, \\
F &= 4a^2 - 6ab - 7b^2,
\end{aligned}$$

find values as follows:

5. $D + E - F$.
6. $D - E + F$.
7. $D + E + F$.
8. $D - (E + F)$.
9. $E - D - F$.
10. $F - D - E$.

REMOVING MARKS OF AGGREGATION

Marks of aggregation are frequently employed to combine into a compound term two or more dissimilar terms, a different form being used to indicate the combination of a portion of an expression already contained in another mark.

The difference of $a+b$ and $c-d$ may be indicated thus: $(a+b)-(c-d)$ and the difference of $a-b$ and $c+d$, thus: $(a-b)-(c+d)$. To indicate the difference of these two differences, each may be inclosed in a bracket, thus:

$$\text{I.}\quad [(a+b)-(c-d)]-[(a-b)-(c+d)].$$

To emphasize the fact that this expression consists of two compound terms, the first is also inclosed in a bracket. The first bracket and the first parenthesis can be omitted in the foregoing without any changes of sign, thus:

$$\text{II.}\quad a+b-(c-d)-[(a-b)-(c+d)],$$

the expression now containing four terms, two of them being compound.

To simplify expressions containing marks of aggregation, it is customary to begin with the removal of the marks.

The second (II) of the foregoing expressions is further simplified by removing the bracket and changing the sign between the two compound terms.

$$\text{III.}\quad a+b-(c-d)-(a-b)+(c+d).$$

The parentheses are then removed, thus

$$\text{IV.}\quad a+b-c+d-a+b+c+d.$$

This is now simplified into

$$\text{V.}\quad 2b+2d.$$

Another plan is to begin with the inmost mark.

The parentheses are first removed from II, thus:

$$\text{III}_a.\quad a+b-c+d-[a+b-c+d],$$

then the bracket, giving the result (IV), which is finally simplified into (V).

WRITTEN EXERCISES

Simplify the following:

1. $3a - \{a - [2b - (4a - \overline{2a - 3b})]\} - (b - a)$.

PROCESS

Removing the vinculum, the given expression

$$3a - \{a - [2b - (4a - \overline{2a - 3b})]\} - (b - a)$$

becomes

$$3a - \{a - [2b - (4a - 2a + 3b)]\} - (b - a).$$

Combining terms, the latter becomes

$$3a - \{a - [2b - (2a + 3b)]\} - (b - a).$$

Removing the parentheses, it becomes

$$3a - \{a - [2b - 2a - 3b]\} - b + a.$$

Combining terms, it becomes

$$3a - \{a - [-2a - b]\} - b + a.$$

Removing the bracket, it becomes

$$3a - \{a + 2a + b\} - b + a.$$

Combining terms, it becomes

$$3a - \{3a + b\} - b + a.$$

Removing the brace, it becomes

$$3a - 3a - b - b + a.$$

Combining terms, it becomes

$a - 2b$. *Ans.*

2. $x + y - [2x - 3y - (x + y)]$.
3. $4 + 5 - [8 - 15 - (4 + 5)]$.
4. $a + x - (a - x) - (2a - 3x)$.
5. $a + x - (a - x - 2a) - 3x$.
6. $a + x - \{a - [x - (2a - 3x)]\}$.
7. $3 + 4 - \{3 - [4 - (6 - 12)]\}$.
8. $a + 2c - \{4a - [3c - 2b - (3c + 4a)] - 2(3a + b)\}$.
9. $m - \{n + [6 - (\overline{2m + 3n} - 7) - 2n] + 8 - 3n - 6\}$.
10. $3x - \{4y - [7 + 2x - (5 - 3y) - (x - 2)]\} + 3y$.

INSERTING TERMS IN A PARENTHESIS

A parenthesis preceded by a plus sign may be used to inclose any terms of an expression without changing the signs of the terms.

Thus, $20 + 15 - 7 - 3$ may be written

$$(a)\ 20 + (15 - 7 - 3),\ \text{or}\ (b)\ 20 + 15 + (-7 - 3).$$

As, however, it is customary to have the first term within a parenthesis positive, (*b*) is changed to the following form:

$$(c)\ 20 + 15 - (7 + 3).$$

In (*c*), the signs of the two terms within the parenthesis are changed, the plus sign of the first term 7 then being omitted.

The last three terms of $ax - by - cx + cy$ may be placed in a parenthesis as follows:

$$ax - (by + cx - cy).$$

The last two terms may be placed in a parenthesis as follows:

$$ax - by - (cx - cy).$$

The terms containing x and those containing y may be placed in parentheses as follows:

$$(ax - cx) - (by - cy).$$

WRITTEN EXERCISES

Rewrite the following expressions placing in a parenthesis the last two terms:

1. $2a + 3b - 4x + 7y.$
2. $3m - 4n - 2a - 5b.$
3. $3x + 7y + 4a - 2b.$
4. $a^3 + 3a^2 - a^3b - 3a^2b.$
5. $x^2 - 2ax - b^2 - a^2.$
6. $2mn - 3n - 2m^2 + 3n^2.$

Rewrite the following expressions placing in one parenthesis the terms containing x and in another the terms containing y:

7. $4ax - 8y - 2x + 4ay.$
8. $-3by - 5ax + 2bx + 3ay.$
9. $2x^2 + 3y - 4x - 2y^2.$
10. $x^3 + 3y^2 - 2x - 4y.$
11. $x^3 - 3y^2 + 2x - 4y.$
12. $m^2x - n^2y - mx - ny.$

ADDITION AND SUBTRACTION OF UNLIKE TERMS

While the sum of unlike terms cannot be written in one simple term, the sum of two or more terms having a common letter or letters can be expressed in a compound term.

In giving the sum of ax and bx, a and b may be considered coefficients of x, and the sum of ax and bx, with regard to x, may be written by prefixing to the common letter, x, the indicated sum of the coefficients inclosed in a parenthesis; thus, $(a+b)x$.

In the same way, the expression $ax+bx+cy+dy$ may be simplified with respect to x and to y by placing it in the form $(a+b)x+(c+d)y$.

When terms are inserted in a parenthesis preceded by a − sign, care must be taken to make the necessary changes in the signs of the terms.

Thus, $ax+bx-cy-dy=(a+b)x-(c+d)y.$

WRITTEN EXERCISES

Rewrite the following expressions, placing in one parenthesis the coefficients of x and in another the coefficients of y.

1. $ax-by-ay+bx.$
2. $ax-2by-ay-2bx.$
3. $2ax+3by-2ay+3bx.$
4. $abx-c^2y-aby+c^2x.$
5. $2x+3ax-4y-3by.$
6. $3a^2x+2by-3a^2y-2bx.$

The sum of the similar compound terms $-4(x+y)$ and $7(x+y)$ is obtained by prefixing to the common term $(x+y)$ the algebraic sum of -4 and 7, the numerical coefficients, the result being $3(x+y)$. When these coefficients are literal, their sum may be indicated. Thus the sum of $a(x+y)$ and $-2b(x+y)$ may be written thus: $(a-2b)(x+y)$.

SIGHT EXERCISES

Indicate in one term the sum of the following:

1. $-5(x+y)$ and $-7(x+y)$.
2. $a(x+y)$ and $b(x+y)$.
3. $2a(x+y)$ and $-b(x+y)$.
4. $m(a+2b)$ and $-n(a+2b)$.
5. $2m(a+2b)$ and $-3m(a+2b)$.
6. $-3c(x-3y)$ and $b(x-3y)$.

Multiplication and Division

MONOMIAL MULTIPLIERS

Since $4a^2b \equiv 4 \cdot a \cdot a \cdot b$, and $5a^3b^2c \equiv 5 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c$, the product of $4a^2b$ and $5a^3b^2c$ is $4 \cdot a \cdot a \cdot b \cdot 5 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c$, of which 4 and 5 are the numerical factors, the literal factor a appearing five times, b appearing three times, and c appearing once. The result is condensed into $20a^5b^3c$.

The letters composing a simple algebraic term are generally placed in alphabetical order.

The product of two monomials contains as factors the product of the numerical coefficients together with all the letters contained in both quantities, each letter being written once with an exponent showing the number of times it appears as a factor.

Note. The coefficient 1 or the exponent 1 is not written. The + sign need not be prefixed to the first term of an expression.

When monomials have like signs, their product is positive; when they have unlike signs, their product is negative.

SIGHT EXERCISES

Give products:

1. $6a^2b \cdot 5a^3b^2$.
2. $-4a^2b(-3a^3)$.
3. $7x^3y^2(-4xy)$.
4. $3a^2b \cdot 2ac$.
5. $4a^2x(-bx^3)$.
6. $2a^3bx \cdot 7aby^3$.
7. $-5a^3y(-7a^2b^4y)$.
8. $6a^5x^3y \cdot 8a^2x$.
9. $2a^3bc \cdot 4ab^2x$.
10. $3x(a^2x + 4bx^2)$.
11. $-2a^3b(b^2c - a^2b)$.
12. $(2x - 4y + 3)3x$.
13. $(7m^2n - 2mn^2)mn$.
14. $(-4y^2 + 3z)2y^3z$.
15. $a^2b(a^3 - 2a^2b - b^3)$.
16. $xy(x - y - 2)$.

Simplify the following:

17. $4(x - 2y + 3) - 3(2x - y - 4)$.
18. $a(x - 2y + 1) - 2a(x + y - 2)$.

MONOMIAL DIVISORS

Deduce the rule for the division of monomials. (*a*) How is the coefficient of the quotient determined? (*b*) How is the exponent of each letter of the quotient determined? (*c*) What letters appear in the quotient? (*d*) How is the sign of the quotient determined?

PRELIMINARY EXERCISES

Give the missing number:

1. $-30\,a^4b^2 \equiv 6\,a^2b \times ?$
2. $-15\,a^3x^2 \equiv -5\,ax \times ?$
3. $27\,m^3n^2o \equiv -9\,mn^2 \times ?$
4. $26\,d^2m^3y \equiv 13\,dm^2 \times ?$
5. $28\,m^2n^4 \equiv 14\,n^3 \times ?$
6. $18\,x^4y^5z^2 \div 3\,x^2y^3z \equiv ?$
7. $4\,a^2x^4 \div 2\,ax^2 \equiv ?$
8. $a^3b^2c^2 \times ? \equiv 11\,a^4b^6c^2.$
9. $19\,x^2y \times ? \equiv 38\,x^3y^2.$
10. $17\,m^2n^3 \times ? \equiv 51\,m^4n^4.$

SIGHT EXERCISES

Give quotients:

1. $\dfrac{-30\,a^4b}{6\,a^2b}.$
2. $\dfrac{-15\,a\,x^2}{-5\,ax}.$
3. $\dfrac{18\,x^4y^5z^2}{3\,x^2y^3z}.$
4. $\dfrac{27\,m^3n^2o}{-9\,mn^2}.$
5. $\dfrac{4\,a^2x^4}{2\,ax^2}.$
6. $\dfrac{11\,a^4b^6c^2}{a^3b^2c^2}.$
7. $-38\,x^3y^2 \div 19\,x^2y.$
8. $28\,m^2n^4 \div 14\,n^3.$
9. $\dfrac{4\,a^3x^2 + 5\,a^2x^3}{a^2x^2}.$
10. $\dfrac{8\,a^2b^5 - 16\,a^3b^2}{8\,a^2b^2}.$
11. $-26\,d^2m^3y \div (-13\,dm^2)$
12. $35\,a^2bc \div 7\,abc.$
13. $\dfrac{8\,a^3 - 12\,a^5 + 16\,a^4}{4\,a^3}.$
14. $\dfrac{12\,x^2y + 24\,xy^2 + 12\,y^3}{4\,y}.$
15. $(4\,a^2x + 12\,a^2x^2 + 16\,a^2x^3) \div 4\,a^2x.$
16. $(6\,x^2y^2 - 18\,x^3y + 24\,xy^3) \div -3\,xy.$
17. $(15\,ax^2 + 25\,axy + 5\,a^2x) \div 5\,ax.$
18. $(9\,m^2n^4 - 18\,m^3n^3 - 27\,m^4n^2) \div 3\,m^2n^2.$

BINOMIAL MULTIPLIERS

Multiply $2a + 3ab + b$ by $a^2 - 2ab$.

PROCESS

$$\begin{array}{llrl}
 & 2a + 3ab + b & & (6) \\
 & \underline{\quad a^2 - 2ab \qquad\qquad} & & (-1) \\
 & 2a^3 + 3a^3b + a^2b & & \\
 & \underline{\qquad\qquad\quad - 4a^2b - 6a^2b^2 - 2ab^2} & & \\
\textit{Ans.} & 2a^3 + 3a^3b - 3a^2b - 6a^2b^2 - 2ab^2 & &
\end{array}$$

The product by a^2 is first written, then the product by $-2ab$, the similar terms being placed in the same column. The two partial products are then combined.

TESTS

A rough test for the correctness of the coefficients in the product may be obtained by making a and b each equal to 1 in the factors and in the product.

When $a \equiv b \equiv 1,\ 2a + 3ab + b \equiv 2 + 3 + 1 \equiv 6,$

$$a^2 - 2ab \equiv 1 - 2 \equiv -1,$$

$$2a^3 + 3a^3b - 3a^2b - 6a^2b^2 - 2ab^2 \equiv 2 + 3 - 3 - 6 - 2 \equiv -6.$$

As the product of the values of the factors, 6 and -1, is -6, which is the value of the product, the latter is probably correct, as far as the coefficients are concerned.

Values must be given the letters that will not make any factor equal to zero. The correctness of the following, $(a + b)(a - b) = a^2 - b^2$, should be tested by giving to b a value different from that given to a; say $a = 1$, $b = 2$. Then,

$$(a + b)(a - b) \equiv (1 + 2)(1 - 2) \equiv 3(-1) \equiv -3.$$
$$(a^2 - b^2) \equiv 1 - 4 \equiv -3.$$

Multiply $2a + 3b$ by $4a + 3c$.

PROCESS

$$\begin{array}{l}
2a + 3b \\
\underline{4a + 3c \qquad\qquad\qquad\qquad} \\
8a^2 + 12ab + 6ac + 9bc \quad \textit{Ans.}
\end{array}$$

As no terms are similar, the product is written in one line.

Test: $(2 + 3)(4 + 3) \equiv 5 \cdot 7 \equiv 35.$

$8 + 12 + 6 + 9 \equiv 35.$

WRITTEN EXERCISES

Find products:

1. $(4a^2-b^2)(a^2-9b^2)$.
2. $(3m-2n)(3m+2n)$.
3. $(3b-c)(2b+4c)$.
4. $(2a+3b)(a-2b)$.
5. $(a+b)(a+b)(a+b)$.
6. $(2a+3b)(a+4c)$.
7. $(2x^2+3y^2)(2x^2-3y^2)$.
8. $(a+2b)(a+2b)$.
9. $(x+y)(x-y)(x^2+y^2)$.
10. $(a-b)(a-b)(a-b)$.
11. $(a^2+ab+b^2)(a-b)(a^3+b^3)$.
12. $(a^2-ab+b^2)(a+b)(a^3-b^3)$.
13. $(4+2x+x^2)(2-x)(8+x^3)$.
14. $(9-3x+x^2)(3+x)(27-x^3)$.
15. $(a+b)(a+c)(a+d)$.

The **degree** of an algebraic term is the sum of the exponents of the literal quantities composing the term.

The terms $6a$, x, $24b$, are each of the **first degree**; $2ab$, c^2, $10xy$, are each of the **second degree**; $4y^3$, $7a^2b$, mn^2, z^3, are each of the **third degree**.

An algebraic expression is said to be **homologous** when each of the terms composing it is of the same degree.

The expression $3a+2b$, $2ab+c^2-xy$, and $4y^3-7a^2b$ are homologous expressions of the first, the second, and the third degree, respectively.

The product of homologous expressions is homologous, the degree of the product being the sum of the degrees of the factors.

Note. In testing the product of homologous expressions, examine the degree of each term composing the product.

16. $(x^2-xy+y^2)(x+y)$.
17. $(x^2+xy+y^2)(x-y)$.
18. $(x^3+x^2y+xy^2+y^3)(x-y)(x^4+y^4)$.
19. $(x+y)(x^2+y^2)(x-y)(x^2-y^2)$.
20. $(3a^2-2ab+4b^2)(4a-2b)$.

Divisor (or Quotient) a Binomial

WRITTEN EXERCISES

Find quotients:

1. $(18b^2 + 8a^2 - 24ab) \div (2a - 3b)$.

Before dividing, the terms of the dividend and those of the divisor should be arranged according to the descending or the ascending powers of one of the letters.

Arranging the given dividend according to the descending powers of a, the first term is $8a^2$, containing the second power of a; the next is $-24ab$, containing the first power of a; the remaining term, $18b^2$, does not contain a.

Process

$$\begin{array}{lll|ll}
\text{Dividend:} & 8a^2 - 24ab + 18b^2 & & 2a - 3b & \text{Divisor} \\ \cline{4-4}
& \underline{8a^2 - 12ab} & (1) & 4a - 6b & \text{Quotient} \\
& \quad -12ab + 18b^2 & (2) & & \\
& \quad \underline{-12ab + 18b^2} & (3) & &
\end{array}$$

Dividing $8a^2$, the first term of the dividend, by $2a$, the first term of the divisor, the first term of the quotient is found to be $4a$. In (1) is shown the product of the divisor by the first term of the quotient. In (2) is shown the term remaining after the subtraction of the first partial product from two terms of the dividend together with the remaining term of the dividend. The second term of the quotient, $-6b$, is obtained by dividing the first term of (2) by the first term of the divisor. The product of the divisor by the second term of the quotient is shown in (3). There being no remainder, the quotient is $4a - 6b$.

Test: When $a \equiv b \equiv 1$, the dividend $\equiv 2$, the divisor $\equiv -1$, the quotient $\equiv -2$.

$$2 \div (-1) = -2.$$

The result may also be tested by multiplying the quotient by the divisor.

2. $(4m^2 - 12mn + 9n^2) \div (2m - 3n)$.
3. $(2x^2 - 28x + 96) \div (2x - 12)$.
4. $(3v^2 - 3v - 60) \div (v + 4)$.

5. $(2a^2 - 11ab + 12b^2) \div (a - 4b)$.

6. $(6y^2 - 2yz - 4z^2) \div (3y + 2z)$.

7. $(2m^2 + 9mn + 9n^2) \div (m + 3n)$.

8. $(12x^2 - 7x - 12) \div (4x + 3)$.

9. $(12 + 25x + 12x^2) \div (3 + 4x)$.

10. $(10a^2 - 37a + 7) \div (2a - 7)$.

11. $(7 - 37y + 10y^2) \div (1 - 5y)$.

12. $(64 - a^6) \div (4 - a^2)$.

Process

$$\begin{array}{llll|l}
64 - & a^6 & & & 4 - a^2 \\
\cline{5-5}
\underline{64 - 16a^2} & & & & 16 + 4a^2 + a^4 \quad \textit{Ans.} \\
& 16a^2 - a^6 & & & \\
& \underline{16a^2 - 4a^4} & & & \\
& & 4a^4 - a^6 & & \\
& & \underline{4a^4 - a^6} & &
\end{array}$$

Although $-16a^2$, the second term of the first partial product, is not similar to a^6, it is, for convenience, written in the same column. The first remainder is obtained by bringing down $-16a^2$ with its sign changed, which, with $-a^6$, from the original dividend, completes the partial dividend.

Test: When $a \equiv 1$, dividend $\equiv 63$, divisor $\equiv 3$, quotient $\equiv 21$.

$$63 \div 3 = 21.$$

13. $(m^6 - n^6) \div (m^3 - n^3)$.

14. $(x^6 - y^6) \div (x^2 - y^2)$.

15. $(x^6 + y^6) \div (x^2 + y^2)$.

16. $(x^6 + 64) \div (x^2 + 4)$.

17. $(a^6 - b^6) \div (a^3 + b^3)$.

18. $(8x^6 + 1) \div (2x^2 + 1)$.

Note in the following quotients the exponents and the signs:

19. $(x^3 - y^3) \div (x - y)$.

20. $(x^4 - y^4) \div (x - y)$.

21. $(x^5 - y^5) \div (x - y)$.

22. $(x^6 - y^6) \div (x - y)$.

23. $(x^7 - y^7) \div (x - y)$.

24. $(x^3 + y^3) \div (x + y)$.

25. $(x^4 - y^4) \div (x + y)$.

26. $(x^5 + y^5) \div (x + y)$.

The difference of like powers of two algebraic numbers is exactly divisible by the difference of the numbers; the sum of like odd powers of two such numbers and the difference of their like even powers are exactly divisible by the sum of the numbers.

27. $(x^6-y^6)\div(x+y)$. 28. $(x^7+y^7)\div(x+y)$.

29. $(x^5-32)\div(x-2)$.

Process

$$
\begin{array}{l|l}
x^5-32 & x-2 \\
\underline{x^5-2x^4} & x^4+2x^3+4x^2+8x+16 \quad \textit{Ans.} \\
\quad 2x^4-32 & \\
\quad \underline{2x^4-4x^3} & \\
\qquad 4x^3-32 & \\
\qquad \underline{4x^3-8x^2} & \\
\qquad\quad 8x^2-32 & \\
\qquad\quad \underline{8x^2-16x} & \\
\qquad\qquad 16x-32 & \\
\qquad\qquad \underline{16x-32} &
\end{array}
$$

Test: When $x\equiv 1$.

$x^5-32\equiv -31$; $x-2\equiv -1$;

Quotient: $1+2+4+8+16\equiv 31$.

$-31\div(-1)\equiv 31$.

The second term (-32) of the dividend may be written in all of the partial dividends, or it may be omitted from all but the last.

30. $(x^4-256)\div(x+4)$. 31. $(8x^3+y^3)\div(2x+y)$.

32. $(x^3-11x^2+37x-35)\div(x-5)$.

33. $(x^5-32)\div(x^4+2x^3+4x^2+8x+16)$.

34. $(m^5-n^5)\div(m-n)$. 35. $(m^5+n^5)\div(m+n)$.

36. $(a^5-b^5)\div(a^4+a^3b+a^2b^2+ab^3+b^4)$.

37. $(16a^4-b^4)\div(2a-b)$. 38. $(27m^6-n^3)\div(3m^2-n)$.

39. $(81y^4-16)\div(27y^3+18y^2+12y+8)$.

40. $(2a^2+4ab+2b^2)\div(a+b)$.

41. $(81y^4-16z^4)\div(9y^2-4z^2)$.

42. $(8b^2-16bc+6c^2)\div(2b-3c)$.

43. $(x^2-16)\div(x-4)$. 44. $(x^3+8)\div(x+2)$.

SPECIAL PRODUCTS

The product of two binomials $(x \pm 6)(x \pm 3)$ having a common literal term without a coefficient, and numerical second terms, is obtained as follows:

1.
$$\begin{array}{rr} & x+6 \\ & x+3 \\ \hline & x^2+6x \\ & 3x+18 \\ \hline \textit{Ans.} & x^2+9x+18 \end{array}$$

2.
$$\begin{array}{rr} & x-6 \\ & x-3 \\ \hline & x^2-6x \\ & -3x+18 \\ \hline \textit{Ans.} & x^2-9x+18 \end{array}$$

3.
$$\begin{array}{rr} & x+6 \\ & x-3 \\ \hline & x^2+6x \\ & -3x-18 \\ \hline \textit{Ans.} & x^2+3x-18 \end{array}$$

4.
$$\begin{array}{rr} & x-6 \\ & x+3 \\ \hline & x^2-6x \\ & 3x-18 \\ \hline \textit{Ans.} & x^2-3x-18 \end{array}$$

An examination of these four results shows that

(1) The first term (x^2) of each result is the square of the common first term of the factors.

(2) The last term of each result is the algebraic product of the second terms of the factors, being + 18 when the second terms of the factors have like signs, and − 18 when the signs are unlike.

(3) The middle term of each result is the common first term of the factors x, with a coefficient equal to the algebraic sum of the second terms of the factors, being $9x$ or $-9x$ when the second terms have like signs and $3x$ or $-3x$ when the signs are unlike.

SIGHT EXERCISES

Give products:

1. $(x+2)(x+3)$.
2. $(x+3)(x+4)$.
3. $(x+2)(x+4)$.
4. $(x+2)(x+5)$.
5. $(x+3)(x+5)$.
6. $(x+4)(x+5)$.
7. $(x+2)(x+6)$.
8. $(x+3)(x+6)$.
9. $(x+4)(x+6)$.
10. $(x+5)(x+6)$.
11. $(x-2)(x-3)$.
12. $(x-3)(x-4)$.
13. $(x-2)(x-4)$.
14. $(x-2)(x-5)$.
15. $(x-3)(x-5)$.
16. $(x-4)(x-5)$.
17. $(x-2)(x-6)$.
18. $(x-3)(x-6)$.
19. $(x-4)(x-6)$.
20. $(x-5)(x-6)$.
21. $(x+2)(x-3)$.
22. $(x-2)(x+3)$.
23. $(x-3)(x+4)$.
24. $(x+3)(x-4)$.

25. $(x+2)(x-4)$.
26. $(x-2)(x+4)$.
27. $(x-2)(x+5)$.
28. $(x+2)(x-5)$.
29. $(x-3)(x+5)$.
30. $(x+3)(x-5)$.
31. $(x-4)(x+5)$.
32. $(x+4)(x-5)$.
33. $(x-2)(x+6)$.
34. $(x+2)(x-6)$.
35. $(x+3)(x-6)$.
36. $(x-3)(x+6)$.
37. $(x-4)(x+6)$.
38. $(x+4)(x-6)$.
39. $(x+5)(x-6)$.
40. $(x-5)(x+6)$.
41. $(x+6)(x-8)$.
42. $(x-7)(x-6)$.
43. $(x+6)(x+9)$.
44. $(x-5)(x+9)$.
45. $(x+1)(x+8)$.
46. $(x-2)(x+1)$.
47. $(x-3)(x-1)$.
48. $(x+4)(x-8)$.
49. $(x+9)(x+6)$.
50. $(x-6)(x+1)$.
51. $(x-7)(x-8)$.
52. $(x+8)(x-9)$.
53. $(x+9)(x+8)$.
54. $(x-8)(x+7)$.
55. $(x-7)(x-9)$.
56. $(x+6)(x-7)$.
57. $(x+5)(x+9)$.
58. $(x-4)(x+1)$.
59. $(x-3)(x-8)$.
60. $(x+2)(x-9)$.

SIGHT EXERCISES

Give quotients:

1. $\frac{x^2+5x+6}{x+3}$.
2. $\frac{x^2+5x+6}{x+2}$.
3. $\frac{x^2-5x+6}{x-3}$.
4. $\frac{x^2-x-6}{x+2}$.
5. $\frac{x^2-11x+18}{x-2}$.
6. $\frac{x^2-7x-18}{x+2}$.
7. $\frac{x^2-11x+24}{x-8}$.
8. $\frac{x^2+14x+45}{x+5}$.
9. $\frac{x^2-x-42}{x-7}$.
10. $\frac{x^2-16x+63}{x-9}$.
11. $\frac{x^2-x-56}{x+7}$.
12. $\frac{x^2+17x+72}{x+8}$.
13. $\frac{x^2-17x+72}{x-9}$.
14. $\frac{x^2+15x+56}{x+7}$.
15. $\frac{x^2-8x+12}{x-2}$.
16. $\frac{x^2+15x+56}{x+8}$.
17. $\frac{x^2-14x+45}{x-9}$.
18. $\frac{x^2-14x+48}{x-8}$.
19. $\frac{x^2-2x-48}{x+6}$.
20. $\frac{x^2-14x+49}{x-7}$.
21. $\frac{x^2+12x+36}{x+6}$.
22. $\frac{x^2-10x+25}{x-5}$.
23. $\frac{x^2+8x+16}{x+4}$.
24. $\frac{x^2+14x+49}{x+7}$.

25. $\frac{x^2-7x-18}{x-9}$.

26. $\frac{x^2-11x+24}{x-8}$.

27. $\frac{x^2-3x-4}{x+1}$.

28. $\frac{x^2-4x-45}{x-9}$.

29. $\frac{x^2-13x+40}{x-8}$.

30. $\frac{x^2+5x-24}{x-3}$.

31. $\frac{x^2-2x-48}{x+6}$.

32. $\frac{x^2+5x+4}{x+1}$.

33. $\frac{x^2+14x+45}{x+9}$.

34. $\frac{x^2-7x-18}{x+2}$.

35. $\frac{x^2-5x-24}{x+3}$.

36. $\frac{x^2-14x+45}{x-5}$.

37. $\frac{x^2+15x+54}{x+6}$.

38. $\frac{x^2-4x-32}{x-8}$.

39. $\frac{x^2-5x-24}{x-8}$.

40. $\frac{x^2-3x-4}{x-4}$.

41. $\frac{x^2+15x+54}{x+6}$.

42. $\frac{x^2-10x+24}{x-4}$.

43. $\frac{x^2-11x+24}{x-3}$.

44. $\frac{x^2+10x+24}{x+6}$.

45. $\frac{x^2-14x+45}{x-9}$.

46. $\frac{x^2+11x+24}{x+3}$.

47. $\frac{x^2+4x-45}{x-5}$.

48. $\frac{x^2-x-72}{x-9}$.

49. $\frac{x^2-18x-63}{x+3}$.

50. $\frac{x^2+5x+4}{x+4}$.

51. $\frac{x^2-x-72}{x+8}$.

52. $\frac{x^2-19x+60}{x-15}$.

53. $\frac{x^2-18x-40}{x+2}$.

54. $\frac{x^2+7x-18}{x-2}$.

55. $\frac{x^2+11x+24}{x+8}$.

56. $\frac{x^2+14x+48}{x+6}$.

57. $\frac{x^2+14x+40}{x+10}$.

58. $\frac{x^2+5x-24}{x+8}$.

59. $\frac{x^2-x-56}{x+7}$.

60. $\frac{x^2+7x-18}{x+9}$.

In the following equations simplify the first member mentally, then give value of x.

61. $\frac{x^2+5x+6}{x+2}=10$.

62. $\frac{x^2+x-6}{x+3}=10$.

63. $\frac{x^2-5x+6}{x-2}=10$.

64. $\frac{x^2-2x-3}{x-3}=10$.

65. $\frac{x^2-x-6}{x+2}=10$.

66. $\frac{x^2+7x+12}{x+3}=10$.

SQUARES OF NUMBERS

The **square** of a quantity is obtained by employing the quantity twice as a factor.

The square of the sum of two numbers is equal to the square of the first + twice the product of the first by the second + the square of the second.

The square of the difference of two numbers is equal to the square of the first − twice the product of the first by the second + the square of the second.

$$(a+5)^2 = (a+5)(a+5) = a^2 + 10\,a + 25,$$
$$(x-7)^2 = (x-7)(x-7) = x^2 - 14\,x + 49.$$

SIGHT EXERCISES

Give the squares of the following:

1. $(2\,a)^2$.
2. $(-3\,b)^2$.
3. $(4\,x)^2$.
4. $(-5\,y)^2$.
5. $(2\,a^2)^2$.
6. $(-3\,b^2)^2$.
7. $(4\,x^3)^2$.
8. $(-5\,y^4)^2$.
9. $(2\,ab)^2$.
10. $(-3\,xy)^2$.
11. $(a+1)^2$.
12. $(b+2)^2$.
13. $(c-1)^2$.
14. $(d-2)^2$.
15. $(x+3)^2$.
16. $(y-4)^2$.
17. $(z-5)^2$.
18. $(m-6)^2$.
19. $(n+7)^2$.
20. $(x+y)^2$.
21. $(x+a)^2$.
22. $(x+2\,a)^2$.
23. $(x+3\,a)^2$.
24. $(x+4\,a)^2$.
25. $(x+5\,a)^2$.
26. $(y-5\,a)^2$.
27. $(y-4\,a)^2$.
28. $(y-3\,a)^2$.
29. $(y-2\,a)^2$.
30. $(y-a)^2$.
31. $(2\,x+a)^2$.
32. $(2\,x-a)^2$.
33. $(2\,x+3\,a)^2$.
34. $(3\,x-2\,a)^2$.
35. $(2\,x+5\,a)^2$.
36. $(5\,x-2\,a)^2$.
37. $(2\,x-7\,a)^2$.
38. $(7\,x+2\,a)^2$.
39. $(3\,x-5\,a)^2$.
40. $(3\,x-7\,a)$.

SQUARES OF ARITHMETICAL NUMBERS

$$73^2 \equiv (70+3)^2 \equiv 70^2 + 2\cdot 3\cdot 70 + 3^2 \equiv 4900 + 420 + 9 \equiv 5329$$
$$69^2 \equiv (70-1)^2 \equiv 70^2 - 2\cdot 1\cdot 70 + 1^2 \equiv 4900 - 140 + 1 \equiv 4761$$

SIGHT EXERCISES

Give the squares of the following:

1. $(21)^2$. 2. $(31)^2$. 3. $(41)^2$. 4. $(51)^2$. 5. $(61)^2$.
6. $(19)^2$. 7. $(29)^2$. 8. $(39)^2$. 9. $(49)^2$. 10. $(59)^2$.
11. $(22)^2$. 12. $(32)^2$. 13. $(42)^2$. 14. $(52)^2$. 15. $(62)^2$.
16. $(71)^2$. 17. $(81)^2$. 18. $(91)^2$. 19. $(74)^2$. 20. $(82)^2$.

SQUARE ROOTS OF NUMBERS

The **square root** of a number is one of the two equal factors of the number.

The sign of the square root is $\sqrt{}$.

$$\sqrt{a^2 + 2ax + x^2} = \pm (a + x). \qquad \sqrt{y^2 + 2y + 1} = \pm (y + 1).$$

As $(+a)(+a) = a^2$ and $(-a)(-a) = a^2$, the square root of a^2 is $\pm a$, the double sign being employed to show that it has two equal roots with opposite signs.

In the following examples give only the positive root.

SIGHT EXERCISES

Give the square roots of the following:

1. $\sqrt{4x^2}$.
2. $\sqrt{9y^2}$.
3. $\sqrt{25m^2}$.
4. $\sqrt{16n^2}$.
5. $\sqrt{36a^2}$.
6. $\sqrt{49b^2}$.
7. $\sqrt{64c^2}$.
8. $\sqrt{81z^2}$.
9. $\sqrt{4x^2y^2}$.
10. $\sqrt{9a^2b^2}$.
11. $\sqrt{x^2 + 2x + 1}$.
12. $\sqrt{x^2 - 4x + 4}$.
13. $\sqrt{a^2 + 6a + 9}$.
14. $\sqrt{b^2 - 2b + 1}$.
15. $\sqrt{c^2 + 4x + 4}$.
16. $\sqrt{y^2 - 8y + 16}$.
17. $\sqrt{z^2 + 10z + 25}$.
18. $\sqrt{x^2 - 18x + 81}$.
19. $\sqrt{y^2 + 16y + 64}$.
20. $\sqrt{m^2 - 14m + 49}$.
21. $\sqrt{a^2 + 2ab + b^2}$.
22. $\sqrt{y^2 - 4ay + 4a^2}$.
23. $\sqrt{a^2 + 4ab + 4b^2}$.
24. $\sqrt{x^2 - 10ax + 25a^2}$.
25. $\sqrt{r^2 + 4cr + 4c^2}$.
26. $\sqrt{z^2 + 6bz + 9b^2}$.

PRODUCT OF SUM AND DIFFERENCE

When two binomials have both terms alike but connected by different signs, the product is a binomial.

The product of the sum of two numbers by their difference is the difference of their squares.

SIGHT EXERCISES

Give products:

1. $(x+4)(x-4)$.
2. $(x+3)(x-3)$.
3. $(x+5)(x-5)$.
4. $(x-6)(x+6)$.
5. $(x+7)(x-7)$.
6. $(x-1)(x+1)$.
7. $(x-9)(x+9)$
8. $(2x-1)(2x+1)$.
9. $(2x-3)(2x+3)$.
10. $(2x-5)(2x+5)$.
11. $(3x-1)(3x+1)$.
12. $(3x+2)(3x-2)$.
13. $(3x-4)(3x+4)$.
14. $(3x-5)(3x+5)$.
15. $(3x-7)(3x+7)$.
16. $(4x+5)(4x-5)$.
17. $(4x-3)(4x+3)$.
18. $(5x+6)(5x-6)$.
19. $(6x-5)(6x+5)$.
20. $(7x+9)(7x-9)$.

Arithmetical Products

SIGHT EXERCISES

Give answers:

1. $(20+2)(20-2)$.
2. $(30+3)(30-3)$.
3. $(40+4)(40-4)$.
4. $(50+5)(50-5)$.
5. $(60+4)(60-4)$.
6. $(70+3)(70-3)$.
7. $(80+2)(80-2)$.
8. $(90+1)(90-1)$.
9. $(80-3)(80+3)$.
10. $(70-4)(70+4)$.
11. $23 \cdot 17$.
12. $32 \cdot 28$.
13. $43 \cdot 37$.
14. $52 \cdot 48$.
15. $63 \cdot 57$.
16. $72 \cdot 68$.
17. $81 \cdot 79$.
18. $92 \cdot 88$.
19. $84 \cdot 76$.
20. $71 \cdot 69$.
21. $24 \cdot 16$.
22. $33 \cdot 27$.
23. $42 \cdot 38$.
24. $53 \cdot 47$.
25. $62 \cdot 58$.
26. $73 \cdot 67$.
27. $82 \cdot 78$.
28. $93 \cdot 87$.
29. $85 \cdot 75$.
30. $75 \cdot 65$.

SIGHT EXERCISES

Give quotients:

1. $\frac{x^2-1}{x+1}.$
2. $\frac{x^2-1}{x-1}.$
3. $\frac{x^2-4}{x+2}.$
4. $\frac{x^2-4}{x-2}.$
5. $\frac{x^2-9}{x+3}.$
6. $\frac{x^2-16}{x-4}.$
7. $\frac{x^2-25}{x+5}.$
8. $\frac{x^2-36}{x-6}.$
9. $\frac{x^2-49}{x+7}.$
10. $\frac{x^2-64}{x-8}.$
11. $\frac{4x^2-1}{2x-1}.$
12. $\frac{4x^2-9}{2x+3}.$
13. $\frac{4x^2-25}{2x-5}.$
14. $\frac{4x^2-49}{2x+7}.$
15. $\frac{4x^2-81}{2x-9}.$
16. $\frac{9x^2-1}{3x+1}.$
17. $\frac{9x^2-4}{3x-2}.$
18. $\frac{9x^2-16}{3x+4}.$
19. $\frac{9x^2-25}{3x-5}.$
20. $\frac{9x^2-49}{3x+7}.$
21. $\frac{1-x^2}{1+x}.$
22. $\frac{1-x^2}{1-x}.$
23. $\frac{4-x^2}{2+x}.$
24. $\frac{4-x^2}{2-x}.$
25. $\frac{9-x^2}{3+x}.$
26. $\frac{16-x^2}{4-x}.$
27. $\frac{25-x^2}{5+x}.$
28. $\frac{36-x^2}{6-x}.$
29. $\frac{49-x^2}{7+x}.$
30. $\frac{64-x^2}{8-x}.$
31. $\frac{81-x^2}{9+x}.$
32. $\frac{4-9x^2}{2-3x}.$
33. $\frac{9-16x^2}{3+4x}.$
34. $\frac{4-25x^2}{2-5x}.$
35. $\frac{9-25x^2}{3+5x}.$
36. $\frac{16-25x^2}{4-5x}.$
37. $\frac{1-36x^2}{1+6x}.$
38. $\frac{25-36x^2}{5-6x}.$
39. $\frac{36-49x}{6+7x}.$
40. $\frac{25-49x^2}{5-7x}.$
41. $\frac{x^2-a^2}{x-a}.$
42. $\frac{x^2-4a^2}{x+2a}.$
43. $\frac{x^2-9a^2}{x-3a}.$
44. $\frac{x^2-16a^2}{x+4a}.$
45. $\frac{4x^2-9a^2}{2x-3a}.$

SPECIAL PRODUCTS OF ARITHMETICAL NUMBERS

The product of two arithmetical numbers may frequently be written at sight by making special mental combinations of the partial products.

Thus, the product of 13 and 18, or 13 times $(10 + 8)$
$\equiv 13$ tens $+ 13$ eights
$\equiv 13$ tens $+ 8$ tens $+ 8$ threes
$\equiv (13 + 8)$ tens $+ 8$ threes
$\equiv 21$ tens $+ 8 \cdot 3 \equiv 210 + 24 \equiv 234$.

That is, the product of two numbers between 11 and 19, inclusive, may be found by taking as tens the first number increased by the ones of the second, and increasing this result by the product of the ones of both numbers.

$$14 \cdot 19 \equiv (14 + 9) \text{ tens} + (4 \cdot 9)$$
$$15 \cdot 15 \equiv (15 + 5) \text{ tens} + (5 \cdot 5)$$

SIGHT EXERCISES

Give products:

1. $14 \cdot 14$.
2. $16 \cdot 16$.
3. $17 \cdot 17$.
4. $18 \cdot 18$.
5. $19 \cdot 19$.
6. $14 \cdot 18$.
7. $13 \cdot 17$.
8. $19 \cdot 13$.
9. $14 \cdot 17$.
10. $19 \cdot 14$.
11. $19 \cdot 16$.
12. $15 \cdot 19$.
13. $19 \cdot 17$.
14. $18 \cdot 15$.
15. $17 \cdot 18$.

The product of any two numbers of two figures each having like tens' digits can be found in this way. The method is, however, limited, with larger numbers, to such as have 10 for the sum of the units' digits.

$$22 \cdot 26 \equiv (22 + 6) \text{ twenties} + 2 \cdot 6$$
$$53 \cdot 58 \equiv (53 + 8) \text{ fifties} + 3 \cdot 8$$
$$85 \cdot 87 \equiv (85 + 7) \text{ eighties} + 5 \cdot 7$$

SIGHT EXERCISES

Give products:

1. $21 \cdot 21$.
2. $21 \cdot 22$.
3. $22 \cdot 22$.
4. $23 \cdot 27$.
5. $24 \cdot 26$.
6. $31 \cdot 31$.

Fractions and Decimals

The product of two mixed numbers having the same integer and the sum of their fractions being 1, is obtained by increasing one integer by 1, multiplying it by the other, and to the product annexing the product of the fractions.

Thus, $7\frac{1}{2} \times 7\frac{1}{2} \equiv 7 \cdot 8 + \frac{1}{2} \cdot \frac{1}{2} \equiv 56\frac{1}{4}$.

SIGHT EXERCISES

Give products:

1. $2\frac{1}{2} \times 2\frac{1}{2}$.
2. $2\frac{1}{3} \times 2\frac{2}{3}$.
3. $2\frac{1}{4} \times 2\frac{3}{4}$.
4. $2\frac{1}{5} \times 2\frac{4}{5}$.
5. $2\frac{2}{5} \times 2\frac{3}{5}$.
6. $3\frac{1}{2} \times 3\frac{1}{2}$.
7. $4\frac{1}{2} \times 4\frac{1}{2}$.
8. $6\frac{1}{2} \times 6\frac{1}{2}$.
9. $8\frac{1}{2} \times 8\frac{1}{2}$.
10. $9\frac{1}{2} \times 9\frac{1}{2}$.
11. $3\frac{1}{4} \times 3\frac{3}{4}$.
12. $4\frac{1}{3} \times 4\frac{2}{3}$.
13. $5\frac{1}{5} \times 5\frac{4}{5}$.
14. $6\frac{2}{5} \times 6\frac{3}{5}$.
15. $7\frac{1}{6} \times 7\frac{5}{6}$.

In the same way give the product of the following:

16. 2.4 × 2.6.
17. 2.2 × 2.8.
18. 2.3 × 2.7.
19. 2.1 × 2.9.
20. 2.5 × 2.5.
21. 3.4 × 3.6.
22. 4.7 × 4.3.
23. 5.2 × 5.8.
24. 6.1 × 6.9.
25. 7.3 × 7.7.
26. 9.2 × 9.8.
27. 8.3 × 8.7.
28. 7.4 × 7.6.
29. 5.5 × 5.5.
30. 4.1 × 4.9.

MISCELLANEOUS SIGHT EXERCISES

Give products:

1. $(70 - 2)(70 + 2)$.
2. $(60 - 3)(60 + 3)$.
3. $(50 + 4)(50 - 4)$.
4. $(40 - 5)(40 + 5)$.
5. $(30 - 6)(30 + 6)$.
6. $(70 + 2)(70 + 8)$.
7. $(60 + 3)(60 + 7)$.
8. $(50 + 4)(50 + 6)$.
9. $(40 + 5)(40 + 5)$.
10. $(10 + \frac{1}{2})(10 + \frac{1}{2})$.
11. $(20 + 2)(20 + 2)$.
12. $(30 - 1)(30 - 1)$.
13. $(20 + 6)(20 + 6)$.
14. $(40 - 1)(40 - 1)$.
15. $(40 + 8)(40 + 8)$.
16. $(50 - 2)(50 - 2)$.
17. $(60 + 3)(60 + 3)$.
18. $(70 - 1)(70 - 1)$.
19. $(10 - \frac{1}{2})(10 - \frac{1}{2})$.
20. $(11 + \frac{1}{2})(11 + \frac{1}{2})$.

Give answers:

1. $11 \div \frac{1}{4}$.
2. $11\frac{1}{4} \div \frac{1}{4}$.
3. $11\frac{1}{2} \div \frac{1}{4}$.
4. $11\frac{3}{4} \div \frac{1}{4}$.
5. $10 \div \frac{1}{8}$.
6. $10\frac{1}{8} \div \frac{1}{8}$.
7. $10\frac{3}{8} \div \frac{1}{8}$.
8. $10\frac{5}{8} \div \frac{1}{8}$.
9. $10\frac{7}{8} \div \frac{1}{8}$.
10. $10\frac{1}{2} \div \frac{1}{8}$.
11. $5\frac{1}{4} \cdot 5\frac{3}{4}$.
12. $5\frac{1}{2} \cdot 5\frac{1}{2}$.
13. $16 \cdot 18$.
14. $73 \cdot 77$.
15. $71 \cdot 69$.
16. $58 \cdot 62$.
17. $102 \cdot 108$.
18. $103 \cdot 106$.
19. $10\frac{1}{4} \cdot 10\frac{3}{4}$.
20. $10\frac{1}{8} \cdot 10\frac{7}{8}$.
21. $444 \times .25$.
22. $484 \times .25$.
23. $488 \times .25$.
24. $844 \times .25$.
25. $884 \times .25$.
26. $.125 \times 48$.
27. $.125 \times 56$.
28. $.125 \times 88$.
29. $.125 \times 96$.
30. $.125 \times 72$.
31. $10.75 \div .25$.
32. $11.25 \div .25$.
33. $21.25 \div .25$.
34. $21.75 \div .25$.
35. $30.25 \div .25$.
36. $30.75 \div .25$.
37. $31.25 \div .25$.
38. $41.75 \div .25$.
39. $51.25 \div .25$.
40. $61.75 \div .25$.
41. $10.125 \div .125$.
42. $10.375 \div .125$.
43. $10.625 \div .125$.
44. $10.875 \div .125$.
45. $20.125 \div .125$.
46. $20.375 \div .125$.
47. $20.625 \div .125$.
48. $20.875 \div .125$.
49. $30.375 \div .125$.
50. $40.625 \div .125$.

Note that $7 \times 48 \times 6 \equiv 48 \cdot 42$.

51. $6 \times 48 \times 7$.
52. $9 \times 56 \times 6$.
53. $7 \times 67 \times 9$.
54. $4 \times 26 \times 6$.
55. $5 \times 35 \times 7$.
56. $9 \times 89 \times 9$.
57. $8 \times 78 \times 9$.
58. $8 \times 66 \times 8$.
59. $7 \times 54 \times 8$.
60. $6 \times 42 \times 8$.
61. $4 \times 42 \times 12$.
62. $7 \times 73 \times 11$.
63. $8 \times 12 \times 94$.
64. $3 \times 31 \times 13$.
65. $2 \times 22 \times 14$.
66. $5 \times 55 \times 11$.

Arithmetical Applications

SIGHT PROBLEMS

1. What is the area of a rectangle whose dimensions are 68 rods by 62 rods?

2. The hypotenuse of a right triangle is 29 feet, and the base is 21 feet. Find the perpendicular.

Find the product of $(29 + 21)$ by $(29 - 21)$. What is the square root of the product?

3. What is the cost of 52 acres of land at $48 per acre?

4. There are 5½ yards in a rod. How many square yards are there in a square rod?

5. What is the product of 16 by 17? At 16½ feet to the rod, how many square feet are there in a square rod?

6. What is the perpendicular of a right triangle whose hypotenuse is 26 feet and whose base is 24 feet? What is its area?

7. What is 54% of $56? Of $560? What is the interest on $560 for 9 years at 6%?

8. At $⅛ per yard, how many yards of muslin can be bought for $10½?

9. What is the cost of 3 pieces of silk, each containing 21 yards, at 67¢ per yard?

10. The parallel sides of a trapezoid measure 40 and 28 rods respectively and the perpendicular distance between them is 36 rods. Find the area.

11. If a kilometer is taken as ⅝ mile, how many kilograms will equal 25 miles?

12. What is the product of 6 times 11 multiplied by 8 times 8?

13. Find the number of cubic inches in a square prism 66 inches high, each side of the base measuring 8 inches.

Miscellaneous Reviews

SIGHT EXERCISES

Give value of x:

1. $7x + 10 = 80.$
2. $90 = 3x + 30.$
3. $4x = 2x + 20.$
4. $7x = 8x - 35.$
5. $x + 3 = b.$
6. $2x + 3 = b.$
7. $cx + 3 = b.$
8. $(a + 3)x = b.$
9. $x + a = b.$
10. $2x + a = b.$
11. $cx + a = b.$
12. $(a + b)x = c.$
13. $19x + 25 = 18x + 30.$
14. $6x - 12 = 7x - 18.$
15. $23 + 5x = 86 - 2x.$
16. $3x - 14 = 14 - 4x.$
17. $10 + 3x = 90 - 7x.$
18. $25 + 3x = 50 - 2x.$
19. $16x - 20 = 17x - 24.$
20. $34 + 15x = 78 + 11x.$
21. $23x - 16 = 20 + 19x.$
22. $60 - 13x = 2x + 30.$

WRITTEN EXERCISES

Find the value of the unknown quantity:

1. $3(x - 1)^2 - 3(x^2 - 1) = x - 15.$
2. $\frac{x + 5}{6} - \frac{x + 1}{9} = \frac{x + 3}{4} - \frac{11}{18}.$
3. $\frac{47}{12} + \frac{x}{4} = \frac{1}{2}\left(4 - \frac{x}{3}\right) + \frac{1}{3}\left(11 - \frac{x}{2}\right).$
4. $\frac{2x - 5}{5} + \frac{11}{10} = \frac{4x - 14}{15} + \frac{4x - 3}{10}.$
5. $\frac{13x + 2}{4} = \frac{2x + 11}{3} + \frac{x + 2}{2}.$
6. $\frac{5x}{3} + \frac{12x + 3}{2} - 17 = \frac{x - 3}{6}.$
7. $\frac{2x - 5}{3} + \frac{5x + 3}{2} = 10x - 48.$
8. $\frac{x + 3}{3} - \frac{x + 2}{2} = \frac{x + 6}{6} - 5.$
9. $\frac{x - 5}{2} + x - 2 = \frac{x - 3}{2} + \frac{x - 4}{3}.$

SIGHT PROBLEMS

1. Find the difference in longitude between two places, one in 70° east longitude and the other in 30° west longitude.

2. Divide $100 between M and N so that one may have $25 more than the other.

3. How many dollars does N receive if he receives $30 less than M and both together receive $100?

4. The difference between two numbers is 20, and the smaller is one sixth of the larger. What are the numbers?

5. The sum of two numbers, one of which is 7 times the other, is 56. Find the numbers.

6. The sum of $\frac{1}{2}$ and $\frac{1}{3}$ of a number is 45. What is the number?

7. A has three times as many horses as B. Together they have 60 horses. How many has each?

8. After spending $\frac{1}{3}$ of her money and $\frac{1}{2}$ of the remainder a girl has 25 cents left. How much money had she before spending any?

9. In how many hours will a boy going 4 miles an hour overtake one going 3 miles per hour and having an hour's start?

10. Divide 70 into two parts having the ratio of 4 to 3.

11. The difference between two numbers bearing the ratio of 1 to $2\frac{1}{2}$ is 12. What are the numbers?

12. Goods marked at $20 are sold at a discount of 10%. At what price should they be marked in order that they may be sold at a discount of 40% without reducing the actual selling price?

13. A man having to go 30 miles in 6 hours, walks at a uniform rate for 3 hours. Then he rides the rest of the way at a rate 4 miles faster per hour, completing his journey on time. How many miles did he walk?

WRITTEN PROBLEMS

1. Divide 121 into two parts one $4\frac{1}{2}$ times the other.

2. The sum of two numbers is 150, and 8 times one number is equal to 7 times the other. What are the numbers?

3. The difference between $\frac{2}{5}$ and $\frac{3}{20}$ of what number is equal to 19?

4. Divide 175 into two parts having the ratio of $2\frac{1}{2}$ to 1.

5. The sum of two numbers is $36\frac{1}{2}$, and their difference is $4\frac{1}{2}$; what are the numbers?

6. The cost of a school library is contributed by 10 persons. If the number of contributors had been increased by 2, the amount paid by each would have been $4 less. Find the cost.

7. After selling $\frac{4}{5}$ of his chickens and $\frac{2}{5}$ of the remainder, a farmer still had 60 chickens. How many had he originally?

8. A man made a deposit of $456 in 1's, 2's, and 5's, the same number of each; how many dollars did he deposit of each denomination?

9. A woman paid the same sum for two packages of tea, one containing 2 pounds more than the other, the respective prices being 40 cents and 50 cents per pound. How many pounds did she buy?

10. The daily wages of three groups of workmen are $2.25, $3.25, and $4.25 per day, respectively, and their daily pay roll is $144.75. How many are there in each group if the first comprises 5 more than the second and 8 more than the third?

11. A 60-gallon cask is filled with 9 gallons of vinegar costing 20 cents per gallon, a quantity of vinegar costing 15 cents per gallon, and water, the mixture costing 14 cents per gallon. How many gallons of water were used?

12. A and B have farms of the same size. A sold $41\frac{1}{2}$ % of his farm and B sold $7\frac{1}{2}$ % of his farm, the former selling 51 acres more than the latter. How many acres did each sell?

13. Two trains starting at noon from points 512 miles apart meet at 8 P.M. How many miles per hour does each travel, if their respective rates of speed are in the ratio of 5 to 3?

14. A dealer regularly makes a discount of 15 % on certain articles marked $ 12. How shall he mark them to obtain the same price after allowing a discount of 20 % from the price?

15. Two pipes can fill a tank in 12 and 20 minutes, respectively, while the waste pipe can discharge 15 gallons per minute. If all the pipes are opened, the tank is filled in 2 hours. How many gallons does it hold?

16. A man paid $60 per acre for a farm. He retained 6 acres, and sold the remainder for $ 75 per acre, receiving for the portion sold 20 % more than he paid for the whole. How many acres did he sell?

17. A, B, and C, running, respectively, at the rate of 120, 80, and 100 yards per minute, start at the same time, B being 1200 yards in advance of A, and C being 600 yards in advance of B. In what time will A be exactly midway between B and C?

In x minutes, A is $120x$ yards from starting point; B, $80x + 1200$; C, $100x + 1800$. Make a diagram.

18. Two bins of the same size are, respectively, one half full of coffee costing 25 cents per pound and one third full of coffee costing 20 cents per pound. One half the contents of the first bin is mixed with the contents of the second bin. One half of this mixture is then placed in the first, making the value of the coffee then in it $15.25. Find the number of pounds each bin will hold.

19. A boy makes a journey of 44 miles in 7 hours. He walks for 3 hours, then rides the rest of the way at a rate 4 miles faster per hour. How many miles per hour does he walk?

20. A train starts from Paris at 7.50 P.M., going at the rate of 50 kilometers per hour. At what time will it meet a train that left Toulouse at 6.35 P.M., going 56 kilometers per hour, the distance between the two cities being 706 kilometers?

Find the distance between the trains at 7.50 P.M. Then add to 7.50 P.M. the time required to reach the meeting point.

CHAPTER II

FACTORING

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Factors

The **factors** of a number are the numbers whose continued product is equal to the given number.

Since $(a+3)(a-7)=a^2-4a-21$, the factors of $a^2-4a-21$ are $(a+3)$ and $(a-7)$.

It is generally unnecessary to separate a monomial into factors.

Monomial Factors of Polynomials

In factoring a polynomial, its terms should first be examined for a common monomial factor. If there be one, the polynomial should be resolved into two factors, one of them being the monomial of the highest degree, and containing the largest arithmetical number.

Thus, the factors of $ab+ac+a^2d$ are a and $(b+c+ad)$; the factors of $4a^3b+6a^2c$ are $2a^2$ and $(2ab+3c)$; that is,

$$ab+ac+a^2d \equiv a(b+c+ad) \qquad 4a^3b+6a^2c \equiv 2a^2(2ab+3c)$$

SIGHT EXERCISES

Give the highest monomial factor of each of the following expressions:

1. $5\,a^2x + 10\,a^2y$.
2. $m^2n + 2\,mn^2$.
3. $8\,c^2d - 18\,d$.
4. $32\,b^2 - 8\,bc$.
5. $14\,x^2 + 35\,x^2y$.
6. $9\,mn + 36\,m^2n^2$.
7. $12\,a^3 - 18\,a^2$.
8. $20\,xy - 45\,y^2$.
9. $24\,ab^3 + 32\,a^2$.
10. $8\,m^2n + 8\,mn^2$.
11. $6\,a^2c + 9\,abc$.
12. $8\,x^2y^2 + 12\,xy$.
13. $18\,acx - 8\,a^2c$.
14. $30\,a^2b^2 - 9\,ab$.
15. $14\,x^4 - 21\,x^2y^2$.
16. $10\,m^2n^2 + 20\,m^4$.
17. $12\,a^3b^2 + 16\,b^4$.
18. $11\,x^4y - 22\,x^2y^3$.
19. $4\,a^2b^3 - 12\,a^3b^2$.
20. $5\,xyz + 15\,x^2z$.
21. $m^3n^3 - 8\,m^2n^2$.
22. $39\,z^3 - 13\,y^2z^2$.
23. $21\,c^2x^2 + 36\,cdx$.
24. $5\,a^3b + 15\,a^2b^2$.
25. $16\,x^2yz - 12\,xy^2z$.
26. $49\,m^4 - 21\,m^2n^2$.
27. $48\,a^3b + 32\,ab^3$.
28. $20\,y^3z + 15\,xyz$.
29. $16\,abc - 8\,a^2b^2c^2$.
30. $14\,m^3n - 35\,m^2n^2$.
31. $5\,a^2x + 10\,a^2y - 10\,a^2m + 20\,a^2n$.
32. $4\,b^2c^4 + 16\,b^3c^3 - 18\,b^4c^3 - 12\,b^3c^4$.
33. $6\,a^3x - 15\,a^2y^2 + 18\,ay^3 - 9\,ax^4$.
34. $12\,m^3n^3 + 16\,m^2n^4 - 20\,m^4n^2 + 8\,mn^5$.

Polynomial Factors of Binomials

Since the product of the sum of two numbers by their difference is the difference of the squares of the numbers,

A binomial that is the difference of the squares of two numbers may be resolved into two factors, one being the sum of the numbers, and the other the difference of the numbers.

$x^2 - y^2 \equiv (x + y)(x - y)$. $a^2 - 4\,b^2 \equiv (a + 2\,b)(a - 2\,b)$.

$16 - n^2 \equiv (4 + n)(4 - n)$. $m^2 - 9 \equiv (m + 3)(m - 3)$.

Difference of Two Squares

SIGHT EXERCISES

Give the two binomial factors of each of the following binomials:

1. x^2-4y^2.
2. $4x^2-y^2$.
3. x^2-4.
4. $4-x^2$.
5. $9-m^2$.
6. m^2-9n^2.
7. n^2-9.
8. $4a^2-9b^2$.
9. c^2-16d^2.
10. $16-z^2$.
11. $4c^2-9$.
12. $9-4z^2$.
13. $9a^2-16b^2$.
14. $16a^2-9$.
15. $9y^2-z^2$.
16. $4m^2-9n^2$.
17. $16x^2-25$.
18. $64-9t^2$.
19. $49-m^2$.
20. $25x^2-y^2$.
21. $25-d^2$.
22. a^2-25.
23. $4a^2-25$.
24. $9b^2-25c^2$.
25. $25-9x^2$.
26. x^2-36a^2.
27. $25a^2-36$.
28. m^2-49n^2.
29. $64-v^2$.
30. x^2-64y^2.
31. $4a^2-49b^2$.
32. $36m^2-1$.
33. $49b^2-64g^2$.
34. $1-81u^2$.
35. $9c^2-49d^2$.
36. $64-9x^2$.
37. $9a^2-64b^2$.
38. $49-16x^2$.
39. $64-25p^2$.
40. $16a^2-49b^2$.
41. $49-4m^2$.
42. $64x^2-9$.
43. $36a^2-49b^2$.
44. $81x^2-1$.
45. $1-64c^2$.
46. $25m^2-49n^2$.
47. $36d^2-1$.
48. $25d^2-64e^2$.
49. $1-100v^2$.
50. $49-9z^2$.
51. $4a^2-81b^2$.

Give answers:

NOTE that $5^2-4^2\equiv(5+4)(5-4)\equiv 9\cdot 1\equiv 9$.

52. 13^2-12^2.
53. 37^2-35^2.
54. 26^2-24^2.
55. 41^2-40^2.
56. 61^2-60^2.
57. 50^2-40^2.

Give the length of the square of the perpendicular in each of the following right triangles:

NOTE that $P^2=H^2-B^2=(H+B)(H-B)$.

58. Hypotenuse, 25; Base, 24.
59. Hypotenuse, 17; Base, 15.

WRITTEN EXERCISES

Write from inspection the factors of the following:

1. $144\,a^2b - 196\,b^3$.

PROCESS

Resolving the given expression into a monomial factor and the remaining binomial factor

$$144\,a^2b - 196\,b^3 \equiv 4\,b(36\,a^2 - 49\,b^2).$$

As the latter is the difference of the squares of $6\,a$ and $7\,b$, respectively, it is resolvable into two factors. The result may be given in this form:

$$144\,a^2b - 196\,b^3 \equiv 4\,b(36\,a^2 - 49\,b^2)$$
$$\equiv 4\,b(6\,a - 7\,b)(6\,a + 7\,b). \quad \textit{Ans.}$$

2. $20\,x^2 - 5\,y^2$.
3. $7\,x^2 - 28$.
4. $90 - 10\,m^2$.
5. $5\,m^2 - 45\,n^2$.
6. $4\,a^3 - 9\,ab^2$.
7. $16\,x - xz^2$.
8. $9\,ab - 4\,abz^2$.
9. $18\,a^2 - 32\,b^2$.
10. $4\,m^2x - 9\,n^2x$.
11. $3\,x^2 - 12\,y^2$.
12. $25\,x^2y - 64\,y^3$.
13. $a - 36\,ad^2$.
14. $28\,c^2 - 63$.
15. $27\,a^2 - 48\,b^2$.
16. $20\,m^2 - 45\,n^2$.
17. $32\,x^2 - 50$.
18. $98 - 2\,m^2$.
19. $3\,x^2 - 75\,y^2$.
20. $12\,a^2x - 75\,x$.
21. $x^3 - 100\,x$.
22. $16\,a^2x - 100\,b^2x$.
23. $27\,b^2c - 75\,c^3$.
24. $50\,a - 18\,ax^2$.
25. $2\,x^2y^2 - 72\,a^2y^2$.
26. $50\,a^3 - 72\,a$.
27. $2\,m^2n - 98\,n^3$.
28. $98\,b^2 - 8\,a^2$.
29. $72\,m^2n^2 - 2\,n^2$.
30. $18\,c^4 - 98\,c^2d^2$.
31. $5\,a^2x - 125\,x$.
32. $20\,a^2b - 405\,b^3$.
33. $147\,z - 27\,z^3$.
34. $600\,v^2w - 24\,w$.
35. $75\,d^2 - 192\,e^2$.
36. $6 - 216\,m^2$.
37. $50\,a^2c - 98\,b^2c$.
38. $144\,m^2 - 196$.
39. $18\,t^2 - 128$.
40. $12\,x^2z - 48\,y^2z$.
41. $36 - 81\,c^2$.
42. $18\,a^2 - 72\,a^2b^2$.
43. $80\,m^2n - 45\,n$.
44. $64\,y - 100\,x^2y$.
45. $192\,a^2x^2 - 27\,b^2y^2$.
46. $256 - 100\,r^2$.
47. $128\,m^2n - 392\,n^3$.

Difference of Two Cubes

SIGHT EXERCISES

Note. The difference of the cubes of two numbers is exactly divisible by the difference of the numbers.

$$(x^3 - y^3) \div (x - y) \equiv x^2 + xy + y^2.$$

Give quotients:

1. $(a^3 - b^3) \div (a - b)$.
2. $(a^3 - 2^3) \div (a - 2)$.
3. $(x^3 - 8) \div (x - 2)$.
4. $(8 - x^3) \div (2 - x)$.
5. $(2^3 - 1^3) \div (2 - 1)$.
6. $(3^3 - 2^3) \div (3 - 2)$.
7. $(4^3 - 2^3) \div (4 - 2)$.
8. $(3^3 - 1^3) \div (3 - 1)$.
9. $(x^3 - 1) \div (x - 1)$.
10. $(1 - x^3) \div (1 - x)$.
11. $(a^3 - b^3) \div (a^2 + ab + b^2)$.
12. $(x^3 - 8) \div (x^2 + 2x + 4)$.
13. $(8 - x^3) \div (4 + 2x + x^2)$.
14. $(2^3 - 1) \div (2^2 + 2 \cdot 1 + 1)$.
15. $(27 - 8) \div (3^2 + 2 \cdot 3 + 2^2)$.
16. $(a^3 - 27) \div (a^2 + 3a + 9)$.
17. $(27 - x^3) \div (9 + 3x + x^2)$.
18. $(m^3 - 64) \div (m^2 + 4x + 16)$.
19. $(64 - m^3) \div (16 + 4x + m^2)$.
20. $(27x^3 - 8y^3) \div (9x^2 + 6xy + 4y^2)$.
21. $(8a^3 - 27) \div (2a - 3)$.
22. $(8x^3 - 27y^3) \div (2x - 3y)$.
23. $(27a^3 - 64b^3) \div (3a - 4b)$.
24. $(64m^3 - 27n^3) \div (4m - 3n)$.
25. $(y^3 - 64) \div (y - 4)$.
26. $(64 - y^3) \div (4 - y)$.
27. $(a^3 - 125) \div (a - 5)$.
28. $(125 - b^3) \div (5 - b)$.
29. $(8a^3 - 125b^3) \div (2a - 5b)$.
30. $(125x^3 - 27y^3) \div (5x - 3y)$.

A binomial composed of the cubes of two numbers connected by a minus sign can be resolved into two factors ; one a binomial and the other a trinomial, the first being the difference of the numbers and the second the sum of the squares of the numbers added to the product of the numbers.

SIGHT EXERCISES

Give the factors:

1. $a^3 - b^3$.
2. $b^3 - a^3$.
3. $x^3 - 8$.
4. $8 - x^3$.
5. $m^3 - 8n^3$.
6. $8n^3 - m^3$.
7. $a^3 - 27$.
8. $27 - a^3$.
9. $x^3 - 27y^3$.
10. $27x^3 - y^3$.
11. $8c^3 - 27d^3$.
12. $27d^3 - 8c^3$.
13. $a^3 - 64$.
14. $64 - a^3$.
15. $x^3 - 64y^3$.
16. $64x^3 - y^3$.
17. $27a^3 - 64b^3$.
18. $27x^3 - 64$.
19. $64 - 27n^3$.
20. $64x^3 - 27y^3$.
21. $a^3 - 125$.

Sum of Two Cubes

As $x^3 + y^3$ is exactly divisible by $x + y$, the quotient being $x^2 - xy + y^2$,

A binomial composed of the sum of the cubes of two numbers can be resolved into two factors, the first being the sum of the numbers, and the other being the sum of the squares of the numbers diminished by their product.

SIGHT EXERCISES

Give the factors:

1. $a^3 + b^3$.
2. $a^3 + 8$.
3. $8 + a^3$.
4. $a^3 + 27$.
5. $27 + a^3$.
6. $x^3 + 64$.
7. $64 + x^3$.
8. $8m^3 + 27$.
9. $27 + 8y^3$.
10. $27a^3 + 64$.
11. $a^3 + 8b^3$.
12. $x^3 + 27y^3$.
13. $m^3 + 64n^3$.
14. $8x^3 + 27y^3$.
15. $27y^3 + 64z^3$.
16. $8a^3 \mp 125$.
17. $27b^3 + 125$.
18. $64c^3 + 125$.
19. $64x^3 + 125y^3$.
20. $27m^3 + 125n^3$.
21. $125a^3 + 8$.

Binomial Factors of Quadratic Trinomials

PRELIMINARY EXERCISES

Give products:

1. $(x+1)(x+18)$.
2. $(x-1)(x-18)$.
3. $(x+1)(x-18)$.
4. $(x-1)(x+18)$.
5. $(x+2)(x+9)$.
6. $(x-2)(x-9)$.
7. $(x+2)(x-9)$.
8. $(x-2)(x+9)$.
9. $(x+3)(x+6)$.
10. $(x-3)(x-6)$.
11. $(x+3)(x-6)$.
12. $(x-3)(x+6)$.

Each of these twelve trinomial products has x^2 for its first term with ± 18 for the last term. The coefficients of the middle term are ± 19, ± 11, ± 9, ± 17, ± 7, ± 3, the first three occurring with 18, and the last three with -18.

13. Find the factors of $x^2+9x+18$.

If the given expression $x^2+9x+18$ is resolvable into two binomial factors, the first term of each must be x, with respective second terms whose product is 18 and whose sum is 9. Since 3 and 6 satisfy these conditions, the required factors are $(x+3)$ and $(x+6)$.

$$x^2+9x+18 \equiv (x+3)(x+6).$$

14. Find the factors of $x^2-9x+18$.

As 18 is positive, the second terms of the required factors have like signs, which must be negative to make the algebraic sum -9. These conditions are satisfied by -3, -6; the required factors are, therefore, $(x-3)(x-6)$.

$$x^2-9x+18 \equiv (x-3)(x-6).$$

15. Find the factors of $x^2+3x-18$.

As the third term of the given trinomial is negative, the second terms of the factors must have contrary signs. In this case their arithmetical difference must be 3, and the larger must be positive in order to make the algebraic sum positive. These conditions are satisfied by -3, $+6$; the required factors are, therefore, $(x-3)(x+6)$.

$$x^2+3x-18 \equiv (x-3)(x+6).$$

16. Find the factors of $x^2-3x-18$.

In this case the larger second term of the factors must be negative since the middle term of the given trinomial is negative. The factors are, therefore, $(x+3)(x-6)$.

$$x^2-3x-18 \equiv (x+3)(x-6).$$

SIGHT EXERCISES

Give the two binomial factors of each of the following trinomials:

1. x^2+7x+6.
2. x^2-7x+6.
3. x^2+5x-6.
4. x^2-5x-6.
5. x^2+5x+6.
6. x^2-5x+6.
7. x^2+x-6.
8. x^2-x-6.
9. $x^2+13x+12$.
10. $x^2-13x+12$.
11. $x^2+11x-12$.
12. $x^2-11x-12$.
13. $x^2+8x+12$.
14. $x^2-8x+12$.
15. $x^2+4x-12$.
16. $x^2-4x-12$.
17. $x^2+7x+12$.
18. $x^2-7x+12$.
19. x^2-x-12.
20. x^2+x-12.
21. $x^2-19x+18$.
22. $x^2-17x-18$.
23. $x^2-7x-18$.
24. $x^2+11x+18$.
25. $x^2+19x+18$.
26. $x^2+17x-18$.
27. $x^2+7x-18$.
28. $x^2-11x+18$.
29. $x^2+25x+24$.
30. $x^2+11x+24$.
31. $x^2+14x+24$.
32. $x^2+5x-24$.
33. $x^2-25x+24$.
34. $x^2-5x-24$.
35. $x^2-14x+24$.
36. $x^2-2x-24$.
37. $x^2-23x-24$.
38. $x^2+2x-24$.
39. $x^2-10x-24$.
40. $x^2-10x+24$.
41. $x^2+23x-24$.
42. $x^2-11x+24$.
43. $x^2+10x+24$.
44. $x^2+31x+30$.
45. $x^2-39x-40$.
46. $x^2+17x+30$.
47. $x^2-14x+40$.
48. $x^2-13x+30$.
49. $x^2+6x-40$.
50. x^2-x-30.
51. $x^2+10x+16$.
52. $x^2+39x-40$.
53. $x^2-18x+32$.
54. $x^2+3x-40$.
55. x^2+x-30.
56. $x^2-6x-40$.
57. $x^2-7x-30$.
58. $x^2-3x-40$.
59. $x^2+13x-30$.
60. $x^2-22x+40$.
61. $x^2+10x-24$.
62. $x^2-31x+30$.
63. $x^2+22x+40$.
64. $x^2-17x+30$.
65. $x^2+41x+40$.
66. $x^2+13x+30$.
67. $x^2+13x+40$.
68. $x^2-11x+30$.
69. $x^2-18x-40$.
70. $x^2-17x+16$.
71. $x^2+29x-30$.
72. $x^2+14x+40$.
73. $x^2-13x-30$.
74. $x^2-13x+40$.
75. $x^2+7x-30$.
76. $x^2+18x-40$.
77. $x^2-12x+32$.
78. $x^2+11x+30$.
79. $x^2-41x+40$.
80. $x^2-29x-30$.
81. $x^2-14x-32$.

WRITTEN EXERCISES

Write from inspection the factors of the following:

1. $5a^2 - 5b^2$.
2. $5a^3 - 5b^3$.
3. $5a^3 + 5b^3$.
4. $7x^2 - 28y^2$.
5. $2x^3 - 16$.
6. $24 - 3x^3$.
7. $5m^3 + 40n^3$.
8. $2a^3 - 54$.
9. $6a^3 - 54a$.
10. $m^3n^3 - 8n^3$.
11. $16m^3n^3 + 2n^3$.
12. $16m^3n^3 - 4mn^5$.
13. $18z^3 - 98z$.
14. $54a^3 + 16$.
15. $56a^3 - 7b^3$.
16. $25d^3 - 64c^2d$.
17. $48x + 6x^4$.
18. $36a^2b - 4a^2bz^2$.
19. $x^2yz - 64y^3z$.
20. $12m^3n + 96n^4$.
21. $3x^2 + 21x + 18$.
22. $2x^2 - 38x + 36$.
23. $4x^2 + 92x - 96$.
24. $3x^2 + 30x - 72$.
25. $2x^2 - 62x + 60$.
26. $2x^2 + 44x + 80$.
27. $3x^2 - 51x + 90$.
28. $2x^2 + 82x + 80$.
29. $3x^2 + 39x + 90$.
30. $2x^2 + 26x + 80$.
31. $3x^2 - 33x + 90$.
32. $5x^2 - 85x + 80$.
33. $3x^2 + 87x - 90$.
34. $3x^2 + 3x - 90$.
35. $2x^2 - 12x - 80$.
36. $3x^2 - 54x + 96$.
37. $2x^2 + 78x - 80$.
38. $5x^2 + 95x + 90$.
39. $8x^2 + 56x + 96$.
40. $7x^2 - 7x + 84$.
41. $8x^3 + 8x^2 - 96x$.
42. $7x^2y - 7xy - 84y$.
43. $6x^3y - 42x^2y + 72xy$.
44. $5ax^2 - 20ax - 60a$.
45. $9x^2y - 9xy - 54y$.
46. $12x^2y^3 - 12x^2y^2 - 72x^2y$.
47. $13m^2n - 65mn + 78n$.
48. $11a^2b + 77ab + 66b$.
49. $4x^2y^3 + 44xy^3 + 96y^3$.
50. $3m^4n^2 + 75m^3n^2 + 72m^2n^2$.
51. $5c^2d^2 - 55cd^2 + 90d^2$.
52. $2x^2 + 12 + 14x$.
53. $15x - 18 + 3x^2$.
54. $2x^2 + 24 + 26x$.
55. $33x + 3x^2 - 36$.
56. $24 + 2x^2 + 16x$.
57. $56x + 96 + 4x^2$.
58. $72 + 33x + 3x^2$.
59. $3x^2y^2 + 9y^2 + 12xy^2$.
60. $9y^2 + 3x^2y^2 - 12xy^2$.
61. $3a^2 + 21a - 90$.

ROOTS OF COMPLETE QUADRATIC EQUATIONS

To find the roots of the equation

$$(a) \qquad x^2 - 5x = -6,$$

it is written in the following form

$$(b) \qquad x^2 - 5x + 6 = 0.$$

Factoring the first member, the equation becomes

$$(c) \qquad (x-3)(x-2) = 0.$$

Making each factor equal to 0, the following equations are obtained.

$$(d)\ x - 3 = 0 \text{ and } (e)\ x - 2 = 0,$$

from which the roots are $x = 3$ and $x = 2$. *Ans.*

Test: When $x \equiv 3$, $(a)\ x^2 - 5x \equiv 9 - 15 \equiv -6$.
When $x \equiv 2$, $x^2 - 5x \equiv 4 - 10 \equiv -6$.

Since the product of the two factors in the first member of (c) is 0, one of the factors must be equal to zero. When $x = 3$, the first factor becomes 0; when $x = 2$, the second factor becomes 0. Either value for x will, therefore, satisfy the given conditions.

This method of finding the roots is applicable to only certain types of quadratic equations.

SIGHT EXERCISES

Give the roots of each of the following:

1. $x^2 + 7x + 6 = 0$.
2. $x^2 - 7x + 6 = 0$.
3. $x^2 - 5x - 6 = 0$.
4. $x^2 + 5x + 6 = 0$.
5. $x^2 - 5x + 6 = 0$.
6. $x^2 + x - 6 = 0$.
7. $x^2 - x - 6 = 0$.
8. $x^2 + 13x + 12 = 0$.
9. $x^2 - 13x + 12 = 0$.
10. $x^2 + 11x - 12 = 0$.
11. $x^2 - 11x - 12 = 0$.
12. $x^2 + 8x + 12 = 0$.
13. $x^2 - 8x + 12 = 0$.
14. $x^2 + 4x - 12 = 0$
15. $x^2 - 4x - 12 = 0$.
16. $x^2 - 29x = 30$.
17. $x^2 - 39x = 40$.
18. $x^2 + 11x = -30$.
19. $x^2 - 12x = -32$.
20. $x^2 + 18x = 40$.
21. $x^2 + 7x = 30$.
22. $x^2 - 13x = 30$.

23. $x^2+29x=30.$

24. $x^2-18x=40.$

25. $x^2+10x=24.$

26. $x^2+13x=30.$

27. $x^2-3x=40.$

28. $x^2-7x=30.$

29. $x^2-6x=40.$

30. $x^2+x=30.$

31. $x^2+11x=-18.$

32. $x^2+19x=-18.$

33. $x^2-11x=-18.$

34. $x^2+25x=-24.$

35. $x^2+11x=-24.$

36. $x^2+14x=-24.$

37. $x^2-25x=-24.$

38. $x^2-14x=-24.$

39. $x^2-10x=-24.$

40. $x^2-11x=-24.$

41. $x^2-14x=-40.$

42. $x^2+10x=-16.$

WRITTEN EXERCISES

Find the values of x:

1. $3x^2-21x=-18.$

SOLUTION

Given $3x^2-21x=-18.$

Transposing, $3x^2-21x+18=0.$

Dividing both members by 3, $x^2-7x+6=0.$

Factoring the first member, $(x-1)(x-6)=0.$

Making each factor equal to 0, $x-1=0;\ x-6=0.$

$\therefore x=1$ or $6.$ *Ans.*

Test: When $x\equiv1$, $3x^2-21x\equiv 3-21\equiv-18.$

When $x\equiv6$, $3x^2-21x\equiv108-126\equiv-18.$

2. $3x^2-15x=18.$

SOLUTION

Given $3x^2-15x=18.$

Transposing and dividing by 3, $x^2-5x-6=0.$

Factoring, $(x+1)(x-6)=0.$

Making each factor equal to 0, etc., $x=-1$ or $6.$ *Ans.*

Test: When $x\equiv-1$, $3x^2-15x\equiv3+15\equiv18.$

$x\equiv6$, $3x^2-15x\equiv108-90\equiv18.$

3. $4x^2 - 48x = -128.$

4. $5x^2 + 90x = 200.$

5. $6x^2 + 42x = 180.$

6. $7x^2 - 91x = -280.$

7. $8x^2 + 56x = 240.$

8. $9x^2 - 63x = -108.$

9. $8x^2 + 88x = -240.$

10. $7x^2 - 91x = 210.$

11. $2x^2 - 58x = 60.$

12. $3x^2 - 123x = -120.$

13. $4x^2 - 68x = -64.$

14. $3x^2 - 54x = 120.$

15. $2x^2 - 22x = -60.$

16. $3x^2 + 39x = -120.$

17. $4x^2 + 52x = -120.$

18. $5x^2 + 205x = -200.$

19. $6x^2 - 102x = -180.$

20. $7x^2 + 154x = -280.$

21. $6x^2 + 84x = -240.$

22. $5x^2 + 145x = 150.$

23. $8x^2 - 176x = -320.$

24. $7x^2 + 91x = 210.$

25. $6x^2 - 18x = 240.$

26. $5x^2 - 35x = 150.$

27. $4x^2 - 24x = 160.$

28. $3x^2 + 3x = 90.$

29. $2x^2 + 6x = 80.$

30. $3x^2 - 54x = -96.$

31. $8x^2 - 248x = -240.$

32. $9x^2 + 90x = 216.$

Binomial Factors of Quadrinomials

PRELIMINARY SIGHT EXERCISES

Give products:

1. $(a + 2)(a - 2).$

2. $(2a + 3)(2a - 3).$

3. $(3a + 2)(3a - 2).$

4. $(4a + 3b)(4a - 3b).$

5. $(3a - 5b)(3a + 5b).$

6. $(a + 2)(a + 2).$

7. $(2a + 3)(2a + 3).$

8. $(3a - 2)(3a - 2).$

9. $(2a + b)(2a + b).$

10. $(a + 2)(a - 3).$

11. $(a + 2)(b + 2).$

12. $(a - 2)(b + 2).$

13. $(a + 2)(b - 2).$

14. $(a + b)(b + c).$

15. $(a - b)(b - c).$

16. $(a - b)(a - c).$

It will be observed that the product of two binomials may be either a binomial, a trinomial, or a quadrinomial.

WRITTEN EXERCISES

Find the factors of the following:

1. $2b - 2a + ab - 4$.

If this quadrinomial is resolvable into factors, one of these must contain a and the other b to produce ab. The terms are therefore rearranged first with respect to a, then with respect to b.

Process

Given $2b - 2a + ab - 4$.

Rearranging terms, $\equiv ab - 2a + 2b - 4$.

Factoring separately the first two and the last two terms,

$\equiv a(b-2) + 2(b-2)$.

Combining the coefficients of the common factor $(b-2)$,

$\equiv (a+2)(b-2)$. *Ans.*

2. $xy + 9 - 3x - 3y$.

Process

$$\begin{aligned} xy + 9 - 3x - 3y &\equiv xy - 3x - 3y + 9. \\ &\equiv (xy - 3x) - (3y - 9). \\ &\equiv x(y-3) - 3(y-3). \\ &\equiv (x-3)(y-3). \quad \textit{Ans.} \end{aligned}$$

Notice the change in the sign of 9 when it is placed in a parenthesis preceded by a minus sign.

3. $ac + ad + bc + bd$.
4. $ac - ad - bc + bd$.
5. $ac + 2ad + 2bc + 4bd$.
6. $ac + ad - 3bc - 3bd$.
7. $ac - 3ad - 3bc + 9bd$.
8. $x^2 + 2ax + 2bx + 4ab$.
9. $x^2 - 2ax - 2bx + 4ab$.
10. $x^3 - x^2 - x + 1$.
11. $x^3 + 3x^2 - 2x - 6$.
12. $x^3 - x^2 - 5x + 5$.
13. $x^4 - x^3 - x + 1$.
14. $x^5 + x^3 - x^2 - 1$.

Review Exercises and Problems

SIGHT EXERCISES

Give products:

1. $(x+y)(x-y)$.
2. $(x^2-y^2)(x^2+y^2)$.
3. $(x^3+y^3)(x^3-y^3)$.
4. $(x^4-y^4)(x^4+y^4)$.
5. $(x^5+y^5)(x^5-y^5)$.
6. $(x^2+xy+y^2)(x-y)$.
7. $(x^4+x^2y^2+y^4)(x^2-y^2)$.
8. $(x^2-xy+y^2)(x+y)$.
9. $(x^4-x^2y^2+y^4)(x^2+y^2)$.
10. $(x^6+x^3y^3+y^6)(x^3-y^3)$.
11. $(x+y)(x-y)(x^2+y^2)$.
12. $(x^2-y^2)(x^2+y^2)(x^4+y^4)$.
13. $(x^3+y^3)(x^3-y^3)(x^6+y^6)$.
14. $(x+y)(x-y)(x^2-y^2)$.
15. $(x^2-y^2)(x^2+y^2)(x^4-y^4)$.

Give quotients:

16. $(x^2-y^2)\div(x-y)$.
17. $(x^3-y^3)\div(x-y)$.
18. $(x^4-y^4)\div(x-y)$.
19. $(x^5-y^5)\div(x-y)$.
20. $(x^6-y^6)\div(x-y)$.
21. $(x^2-y^2)\div(x+y)$.
22. $(x^3+y^3)\div(x+y)$.
23. $(x^4-y^4)\div(x^2-y^2)$.
24. $(x^4-y^4)\div(x^2+y^2)$.
25. $(x^6-y^6)\div(x^3-y^3)$.
26. $(x^6-y^6)\div(x^3+y^3)$.
27. $(x^6-y^6)\div(x^2-y^2)$.
28. $(x^6-y^6)\div(x^2+y^2)$.
29. $(x^8-y^8)\div(x^4-y^4)$.

Give value of x:

30. $\frac{3}{x}=\frac{x}{12}$.
31. $\frac{x}{9}=\frac{4}{x}$.
32. $\frac{x-1}{4}=\frac{2}{x+1}$.
33. $\frac{4}{x+2}=\frac{x-2}{8}$.
34. $x+\frac{x}{2}+\frac{x}{4}=28$.
35. $\frac{x}{2}+\frac{x}{4}+\frac{x}{8}=7$.

Solve the following:

36. $3:x::x:12$.
37. $x:9::4:x$.
38. $3x+3=x+15$.
39. $12-5x=3x-4$.
40. $2:x+1::x-1:4$.
41. $x-2:3::7:x+2$.
42. $6(x-1)-5(x-1)=1$.
43. $4(x-3)+2(x-3)=6$.

SIGHT PROBLEMS

1. Find the value of x.

(*a*) When $(3+x)$ multiplied by $(3-x)=5$.

(*b*) When $(x+4)$ times $(x-4)=9$.

(*c*) When $(x-2)(x^2+2x+4)=19$.

2. Show (*a*) that the product of 74 by 76 is equal to $(80 \cdot 70)+(4 \cdot 6)$.

(*b*) That $75 \cdot 75=(80 \cdot 70)+5^2$.

(*c*) That $7\frac{1}{2} \times 7\frac{1}{2}=(8 \cdot 7)+(\frac{1}{2})^2$.

3. (*a*) What are two factors of 5621?

(*b*) Give the prime factors.

4. (*a*) What is the square root of 4225?

(*b*) Of $42\frac{1}{4}$?

(*c*) What two numbers between 60 and 70 give a product of 4224?

(*d*) Give the value of x in $(x+2)(x+8)=4216$.

5. The product of two numbers is 4209 and their difference is 8. What are the numbers?

6. Divide $300 into two parts that shall have the ratio of 7 : 8.

7. A, B, and C divide profits of $300, A's share being to B's as 2 to 3, and A's share being to C's as 2 to 5. What is the share of each?

8. Find three numbers whose sum is 50, the first being to the second as 2 to 3, and the second to the third as 3 to 5.

9. By selling cows for as many dollars each as the number of cows he sold, a farmer received the sum required to pay a certain debt. How many dollars more or less would he have received if he sold one cow fewer and received one dollar apiece more?

10. (*a*) What is the difference between 50 times 50 and 49 times 51?

(*b*) Between 40 times 40 and 38 times 42?

(*c*) Between 57 times 63 and 60 times 60?

11. What is the difference between the positive value of x in the equation $x = \sqrt{144} + \sqrt{25}$ and its positive value in the equation $x = \sqrt{144 + 25}$?

12. Give the difference between the square of the sum of x and y and the sum of the squares of x and y.

13. Give the difference between the sum of the positive square roots of 9 and 16 and the positive square root of the sum of 9 and 16.

14. Show that the difference of the squares of 41 and 40 is equal to the sum of 41 and 40 multiplied by the difference of 41 and 40.

15. (*a*) What is the difference of the squares of 61 and 60? (*b*) Give the value of $\sqrt{61^2 - 60^2}$. (*c*) What is the base of a right triangle whose hypotenuse is 61 feet and whose perpendicular is 60 feet? (*d*) What is the base of one whose hypotenuse is 26, perpendicular 24?

WRITTEN EXERCISES

Find values of x:

NOTE. First square each member, by removing the radical sign.

1. $\sqrt{x^2 - 7x + 17} = \sqrt{5}$.

2. $\sqrt{x^2 - 12x + 43} = \sqrt{8}$.

3. $\sqrt{x^2 - 12x + 17} = \sqrt{6}$.

4. $\sqrt{x^2 - 14x + 51} = \sqrt{3}$.

5. $\sqrt{x^2 - 14x + 31} = \sqrt{7}$.

6. $\sqrt{x^2 - x - 13} = \sqrt{-1}$.

7. $\sqrt{x^2 + x - 6} = \sqrt{-4}$.

8. $\sqrt{x^2 + x - 81} = \sqrt{-9}$.

9. $\sqrt{x^2 - x - 46} = \sqrt{-16}$.

10. $\sqrt{x^2 + x - 81} = \sqrt{-25}$.

Note. First extract square root of each member. Use only the positive root of first member and both roots of second member.

11. $\sqrt{x^2+2x+1}=\sqrt{9}$.
12. $\sqrt{x^2-16x+64}=\sqrt{289}$.
13. $\sqrt{x^2+10x+25}=\sqrt{121}$.
14. $\sqrt{x^2-12x+36}=\sqrt{169}$.
15. $\sqrt{x^2+6x+9}=\sqrt{49}$.
16. $\sqrt{x^2-20x+100}=\sqrt{441}$.
17. $\sqrt{x^2-8x+16}=\sqrt{81}$.
18. $\sqrt{x^2+14x+49}=\sqrt{225}$.
19. $\sqrt{x^2+18x+81}=\sqrt{361}$.
20. $\sqrt{x^2-4x+4}=\sqrt{25}$.

Find value of x:

Note. Use only the positive roots of the unknown numbers.

21. $\sqrt{64x^2}+\sqrt{225x^2}=391$.
22. $\sqrt{64x^2+225x^2}=391$.
23. $\sqrt{25x^2}+\sqrt{144x^2}=221$.
24. $\sqrt{25x^2+144x^2}=221$.
25. $\sqrt{81x^2}+\sqrt{1600x^2}=2009$.
26. $\sqrt{49x^2}+\sqrt{576x^2}=775$.
27. $\sqrt{49x^2+576x^2}=775$.
28. $\sqrt{36x^2}+\sqrt{64x^2}=140$.
29. $\sqrt{36x^2+64x^2}=140$.
30. $\sqrt{81x^2+1600x^2}=2009$.

WRITTEN PROBLEMS

1. The difference of two numbers is 14 and their product is 120. What are the numbers?

$$x(x+14)=120.$$

2. The entire surface of a square prism is 312 square feet, its height being 10 feet and each side of the base being x feet. Find the value of x.

3. What are the sides of a right triangle measuring, respectively, x feet, $x+7$ feet, and $x+9$ feet?

$$(x)^2+(x+7)^2=(x+9)^2.$$

4. What are the dimensions of a rectangle containing 1728 square rods when its sides bear the ratio of 3 to 4?

5. A rectangle having one side 6 yards longer than the adjacent side has an area of 216 square yards. Find the dimensions.

6. The difference of the squares of two consecutive numbers is 49. What are the numbers?

7. The base of a right triangle is 11; the numbers representing the lengths of the other sides are consecutive. Find the lengths of the other sides.

8. Twenty per cent of a number multiplied by thirty per cent of the same number is 216. What is the number?

9. Twenty per cent of thirty per cent of a number is 216. What is the number?

10. What two consecutive even numbers have 340 for the sum of their squares?

11. What two consecutive odd numbers have a product of 255?

$$(x-1)(x+1)=255.$$

12. The product of two numbers having the ratio of 7 to 9 is 1008. Find the numbers.

13. What two numbers in the ratio of 4 to 5 have 1296 for the difference of their squares?

14. Two numbers are to each other as 17 to 23. When 5 is added to each, the ratio of the respective sums is 3 to 4. What are the numbers?

15. When 2, 4, 6, and 12 are, respectively, added to four consecutive even numbers, the respective sums constitute a proportion. Find the numbers.

CHAPTER III

FRACTIONS

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Algebraic Fractions

A fraction is the indicated division of one number by another, the dividend being written above a horizontal line, and the divisor below it, the former being called the numerator, and the latter the denominator.

To distinguish between the various forms in which division is indicated, the expression $\frac{a^2y}{b^2x}$ may be read "a^2y over b^2x," $a^2y \div b^2x$ being read "a^2y divided by b^2x," and $a^2y : b^2x$ being read "the ratio of a^2y to b^2x."

An **algebraic fraction** is one that has an algebraic expression for either the numerator or the denominator.

$$\frac{x+1}{3},\ \frac{7}{x+2},\ \frac{x+1}{y-3} \text{ are algebraic fractions.}$$

Some authorities call a fraction "algebraic" only when it has an algebraic expression for the denominator such as the last two; while others limit the term to such as have algebraic expressions in both the numerator and the denominator, such as the last.

Transformations

In operating with fractions it is sometimes necessary to change their form without changing their value. Such a change is called a **transformation.** An algebraic expression, fractional or otherwise, that is transformed into one containing the fewest and the lowest terms is said to be **simplified.**

SIGN OF THE FRACTION

When a fraction is employed to indicate the division of one number by another, the terms of the fraction, if monomials, are generally written without signs, the sign of the quotient being prefixed to the fraction as a whole.

Thus, $4a \div 5b$ or $-4a \div (-5b)$ are both expressed as $\frac{4a}{5b}$, the quotient being positive when the divisor and the dividend have like signs.

When the divisor and the dividend have unlike signs, the minus sign is prefixed to the fraction, as a whole.

Thus, $4a \div (-5b)$ or $-4a \div 5b$ are both expressed as $-\frac{4a}{5b}$, the sign of the fraction being $-$.

That is, $\frac{-4a}{-5b} \equiv \frac{4a}{5b}$; $\frac{-4a}{5b} \equiv -\frac{4a}{5b}$; $\frac{4a}{-5b} \equiv -\frac{4a}{5b}$.

When either the numerator or the denominator is a polynomial having a negative first term, care must be taken to change all the signs of such polynomial.

$$\frac{-a-b}{c} \equiv \frac{-(a+b)}{c} \equiv -\frac{a+b}{c}; \quad \frac{-a-b}{-c} \equiv \frac{-(a+b)}{-c} \equiv \frac{a+b}{c};$$

$$\frac{-x+y}{-m-n} \equiv \frac{-(x-y)}{-(m+n)} \equiv \frac{x-y}{m+n}.$$

SIMPLIFYING A FRACTION

As in arithmetic, the value of a fraction is not changed when both the numerator and the denominator are either multiplied or divided by the same number.

A fraction is **simplified** by rejecting from the numerator and the denominator all the factors common to both. It is then said to be expressed in its **lowest terms.**

To simplify $\frac{15\,a^2bc^2}{25\,a^3b^2cd}$, the numerator and the denominator are examined to determine the common factors. The greatest common divisor of 15 and 25 is 5; the highest common factor of the literal terms is the product of the common letters each with the largest exponent it has in both terms, in this case a^2bc. Dividing both terms of the fraction by $5\,a^2bc$,

$$\frac{15\,a^2bc^2 \div 5\,a^2bc}{25\,a^3b^2cd \div 5\,a^2bc} \equiv \frac{3\,c}{5\,abd}. \quad \textit{Ans.}$$

When either term of a fraction is a monomial, the highest common factor is a monomial.

$\frac{15\,a^2bc^2 - 25\,a^3b^2cd}{10\,ab^3d}$ is simplified by dividing both terms by $5\,ab$, 5 being the greatest common divisor of 15, 25, and 10; and ab being the highest common factor of the letters.

$$\frac{(15\,a^2bc^2 - 25\,a^3b^2cd) \div 5\,ab}{10\,ab^3d \div 5\,ab} \equiv \frac{3\,ac^2 - 5\,a^2bcd}{2\,b^2d}. \quad \textit{Ans.}$$

SIGHT EXERCISES

Simplify the following fractions:

1. $\frac{5\,a^2x}{10\,a^2y}$.
2. $\frac{m^2n}{2\,mn^2}$.
3. $\frac{8\,c^2d}{18\,d}$.
4. $\frac{8\,bc}{32\,b^2}$.
5. $\frac{14\,x^2}{35\,x^2y}$.
6. $\frac{9\,mn}{36\,m^2n^2}$.
7. $\frac{12\,a^2}{18\,a^3}$.
8. $\frac{20\,xy}{45\,y^2}$.
9. $\frac{24\,ab^3}{32\,a^2}$.
10. $\frac{8\,m^2n}{8\,mn^2}$.
11. $\frac{6\,a^2c}{9\,abc}$.
12. $\frac{8\,x^2y^2}{12\,xy}$.
13. $\frac{18\,acx}{8\,a^2c}$.
14. $\frac{9\,ab}{30\,a^2b^2}$.
15. $\frac{14\,x^4}{21\,x^2y^2}$.
16. $\frac{10\,m^2n^2}{20\,m^4}$.
17. $\frac{12\,a^3b^2}{16\,b^4}$.
18. $\frac{11\,x^4y}{22\,x^2y^3}$.
19. $\frac{4\,a^2b^3}{12\,a^3b^2}$.
20. $\frac{5\,xyz}{15\,x^2z}$.
21. $\frac{m^3n^3}{8\,m^2n^2}$.
22. $\frac{13\,y^2z^2}{39\,z^3}$.
23. $\frac{21\,c^2x^2}{36\,cdx}$.
24. $\frac{5\,a^3b}{15\,a^2b^2}$.
25. $\frac{12\,xy^2z}{16\,x^2yz}$.

WRITTEN EXERCISES

Simplify the following fractions by inspection:

1. $\frac{5\,a^2x + 10\,a^2y}{15\,a^2b^2}$.

2. $\frac{4\,m^2n^2}{m^2n + 2\,mn^2}$.

3. $\frac{8\,c^2d - 18\,d}{4\,c^2d^2}$.

4. $\frac{8\,a^2b}{8\,bc - 32\,b^2}$.

5. $\frac{14\,x^2 + 35\,x^2y}{21\,xy^2}$.

6. $\frac{18\,m^3n^3}{9\,mn - 36\,m^2n^2}$.

7. $\frac{12\,a^2 + 18\,a^3}{30\,a^4}$.

8. $\frac{30\,x^2y^2}{20\,xy + 45\,y^2}$.

9. $\frac{24\,ab^3 - 32\,a^2}{16\,a^2b^2}$.

10. $\frac{4\,m^2n^2}{8\,m^2n - 8\,mn^2}$.

11. $\frac{6\,a^2c + 9\,abc}{12\,abcd}$.

12. $\frac{20\,xyz}{8\,x^2y^2 - 12\,xy}$.

13. $\frac{8\,a^2c + 18\,acx}{12\,a^2c^2x}$.

14. $\frac{24\,a^2bc^2}{9\,ab - 30\,a^2b^2}$.

15. $\frac{14\,x^4 - 21\,x^2y^2}{7\,x^3y^3}$.

SUPPLEMENTARY EXERCISES

16. Simplify $\frac{x^2 - 4}{x^2 + 5\,x + 6}$.

PROCESS

$$\frac{x^2 - 4}{x^2 + 5\,x + 6} \equiv \frac{(x + 2)(x - 2)}{(x + 2)(x + 3)} \equiv \frac{x - 2}{x + 3}. \quad \textit{Ans.}$$

Test: When $x = 1$, the original fraction becomes

$$\frac{x^2 - 4}{x^2 + 5x + 6} \equiv \frac{1 - 4}{1 + 5 + 6} \equiv \frac{-3}{12} = -\frac{1}{4}.$$

The answer becomes

$$\frac{1 - 2}{1 + 3} \equiv \frac{-1}{4} \equiv -\frac{1}{4}.$$

17. $\dfrac{x^2-4}{x^2-5x+6}$.

18. $\dfrac{x^2-x-6}{x^2-4}$.

19. $\dfrac{x^2-9}{x^2+5x+6}$.

20. $\dfrac{x^2-x-6}{x^2-9}$.

21. $\dfrac{x^2-16}{x^2-9x+20}$.

22. $\dfrac{x^2+7x+12}{x^2-16}$.

23. $\dfrac{x^2-25}{x^2+9x+20}$.

24. $\dfrac{x^2+9x+18}{x^2-36}$.

25. $\dfrac{x^2-36}{x^2-3x-18}$.

26. $\dfrac{x^2+7x+10}{x^2-25}$.

27. $\dfrac{x^2-49}{x^2+9x+14}$.

28. $\dfrac{x^2-5x+6}{x^2+x-12}$.

29. $\dfrac{x^2+6x+8}{x^2-x-20}$.

30. $\dfrac{x^2-x-12}{x^2-7x+12}$.

31. $\dfrac{x^2-4x-12}{x^2+9x+14}$.

32. $\dfrac{x^2+9x+18}{x^2-x-42}$.

33. $\dfrac{x^2-x-6}{x^2-8x+15}$.

34. $\dfrac{x^2+3x-10}{x^2+x-20}$.

35. $\dfrac{x^2-9x+14}{x^2-10x+21}$.

36. $\dfrac{x^2-4x-21}{x^2-2x-15}$.

37. $\dfrac{x^2+7x-18}{x^2-8x+12}$.

38. $\dfrac{x^2+9x+20}{x^2-2x-24}$.

39. $\dfrac{x^2-4x-21}{x^2-5x-24}$.

40. $\dfrac{x^2-x-12}{x^2-3x-18}$.

41. $\dfrac{x^2+6x+9}{x^2-2x-15}$.

42. $\dfrac{x^2-4x+4}{x^2-x-2}$.

43. $\dfrac{x^2-2x-3}{x^2+6x+5}$.

44. $\dfrac{x^2-10x+16}{x^2+9x-22}$.

45. $\dfrac{x^2-6x-27}{x^2+4x+3}$.

46. $\dfrac{x^2-9x+20}{x^2+5x-36}$.

47. $\dfrac{x^2+11x+30}{x^2-2x-35}$.

48. $\dfrac{x^2+5x-24}{x^2-13x+30}$.

49. $\dfrac{x^2-6x-40}{x^2+15x+44}$.

CHANGING A FRACTION TO AN ENTIRE NUMBER OR TO A MIXED NUMBER

A fraction is changed to an entire or a mixed number by performing the indicated division, placing the remainder (if any) over the denominator of the fractional part, and connecting it with the entire number by the proper sign.

Thus, $$\frac{15x^2+25x}{3}=5x^2+8x+\frac{x}{3}.$$

SIGHT EXERCISES

Change to entire or to mixed numbers:

1. $\dfrac{25}{8}$.

2. $\dfrac{8a+25}{8}$.

3. $\dfrac{25a^2+8}{8a}$.

4. $\dfrac{25a^2+9a}{8a}$.

5. $\dfrac{25a^2-9a}{8a}$.

6. $\dfrac{x^2-y^2}{x-y}$.

7. $\dfrac{x^2-y^2+y}{x-y}$.

8. $\dfrac{x^2-y^2+y}{x+y}$.

9. $\dfrac{x^2-y^2+y^2}{x-y}$.

10. $\dfrac{x^2-2y^2}{x+y}$.

11. $\dfrac{x^2-5x+6}{x-3}$.

12. $\dfrac{x^2-5x+7}{x-3}$.

13. $\dfrac{x^2+6x+6}{x+3}$.

14. $\dfrac{x^2+7x+6}{x+3}$.

15. $\dfrac{x^2+4x+6}{x+3}$.

WRITTEN EXERCISES

Change to entire or to mixed expressions:

1. $\dfrac{2x^2-5x+6}{x+3}$.

$$\begin{array}{r|l} 2x^2 - 5x + 6 & x+3 \\ \cline{2-2} 2x^2 + 6x \phantom{{}+6} & 2x - 11 + \dfrac{39}{x+3} \\ \cline{1-1} -11x + 6 & \\ -11x - 33 & \\ \cline{1-1} \text{Rem.} \qquad 39 & \end{array}$$

2. $\dfrac{a^3+2b^3}{a^2-ab+b^2}$.

3. $\dfrac{x^3-67}{x^2+4x+16}$.

4. $\dfrac{9a^3+126}{4a^2-10a+25}$.

5. $\dfrac{3x^2+22x+19}{x+1}$.

6. $\dfrac{4x^2-39x+36}{2x^2-36}$.

7. $\dfrac{6x^2-103x+91}{2x-30}$.

8. $\dfrac{8x^3+8x^2-97x}{x-4}$.

9. $\dfrac{6ax^2-20ax-60a}{3x-18}$.

10. $\dfrac{15x^2-9x+4}{3x-9}$.

11. $\dfrac{8x^2-10x-5}{4x-7}$.

ADDITION AND SUBTRACTION

Fractions having the same denominator are combined by placing the algebraic sum of their numerators over the common denominator.

When the fractions have different denominators, they must be changed to equivalent fractions having a common denominator before being added or subtracted.

SIGHT EXERCISES

Give results in lowest terms:

1. $\frac{1}{2}+\frac{1}{3}$.
2. $\frac{a}{2}+\frac{a}{3}$.
3. $\frac{x}{2}+\frac{x}{6}$.
4. $\frac{a}{b}+\frac{a}{2b}$.
5. $\frac{3a}{2b}+\frac{a}{2b}$.
6. $\frac{a}{9}+\frac{a}{12}$.
7. $\frac{a}{x}+\frac{a}{3x}$.
8. $\frac{3a}{2b}-\frac{a}{2b}$.
9. $\frac{a}{b}-\frac{a}{2b}$.
10. $\frac{a}{9}-\frac{a}{12}$.
11. $\frac{a}{x-y}+\frac{3a}{x-y}$.
12. $\frac{x-y}{a}+\frac{x+y}{a}$.
13. $\frac{x+y}{b}-\frac{x-y}{b}$.
14. $\frac{2x}{2}+\frac{3x}{5}$.
15. $x-\frac{3x}{10}$.

The mixed quantity, $x-\frac{3x}{10}$, is transformed into the fraction $\frac{7x}{10}$ by performing the indicated subtraction.

WRITTEN EXERCISES

Simplify the following:

1. $\frac{a-2b}{3a}-\frac{2a-b}{2a}$.

PROCESS

$$\frac{a-2b}{3a}-\frac{2a-b}{2a}\equiv\frac{2(a-2b)-3(2a-b)}{6a}\equiv\frac{2a-4b-6a+3b}{6a}$$

$$\equiv\frac{-4a-b}{6a}\equiv-\frac{4a+b}{6a}.\quad \textit{Ans.}$$

The common denominator is $6a$. Since the common denominator contains the denominator of the first fraction 2 times, the numerator of the first fraction is multiplied by 2 and written above the common denominator. Since the common denominator, $6a$, contains the denominator of the second fraction 3 times, the numerator of the second is multiplied by 3 and written above the common denominator, being placed in a parenthesis preceded by the minus sign. When the parentheses are removed and the terms are collected, the numerator is $-4a-b$. As it is customary to have the first term of the numerator of a fraction positive, the numerator is changed to $4a+b$, and the sign of the fraction made minus.

The answer may be tested by substituting 1 for a and for b, the given expression, thus becoming

$$\frac{1-2}{3}-\frac{2-1}{2}\equiv\frac{-1}{3}-\frac{1}{2}\equiv-\frac{1}{3}-\frac{1}{2}\equiv-\frac{5}{6}.$$

The answer becomes $$-\frac{4+1}{6}\equiv-\frac{5}{6}.$$

2. $3a+\dfrac{a^2-2b^2}{3a}.$

$$3a+\frac{a^2-2b^2}{3a}\equiv\frac{9a^2+a^2-2b^2}{3a}\equiv\frac{10a^2-2b^2}{3a}.\quad \textit{Ans.}$$

The denominator of the term $3a$ is assumed to be 1.

3. $\dfrac{4a-3}{5}-\dfrac{2a+1}{10}.$

4. $\dfrac{x+1}{3}+\dfrac{x-1}{4}.$

5. $\dfrac{x+2}{2}-\dfrac{x-3}{3}.$

6. $a-\dfrac{4a-3}{5}.$

7. $\dfrac{x}{2a}+\dfrac{x}{3a}+\dfrac{x}{6a}.$

8. $\dfrac{x-y}{2}-\dfrac{x-y}{4}.$

9. $\dfrac{x+y}{2}-\dfrac{3x-4y}{7}.$

10. $\dfrac{x-2}{3}+\dfrac{5x-7}{2}+1.$

11. $\dfrac{4a-3}{6}-\dfrac{7a+1}{8}-1.$

12. $\dfrac{4a-3}{5}-\dfrac{7a+1}{4}+a.$

13. $x+\dfrac{x}{3}-\dfrac{x}{4}+\dfrac{x}{5}-\dfrac{x}{6}.$

14. $a+\dfrac{a-b}{4}-\dfrac{2a-b}{2}.$

15. $x+\dfrac{x+y}{2}-y.$

16. $x-\dfrac{3x-4y}{7}+2y.$

17. $\dfrac{1}{x^2-1}+\dfrac{1}{x+1}$.

PROCESS

$$\frac{1}{x^2-1}+\frac{1}{x+1}\equiv\frac{1}{x^2-1}+\frac{x-1}{x^2-1}\equiv\frac{1+x-1}{x^2-1}\equiv\frac{x}{x^2-1}.\quad \textit{Ans.}$$

The lowest common denominator is x^2-1.

The two fractions are rewritten, the first fraction in its original form and the second one transformed into the equivalent $\dfrac{x-1}{x^2-1}$, which is obtained by dividing x^2-1, the common denominator, by $x+1$, the original denominator.

In the next step, both numerators are written over the common denominator.

The numerator is finally simplified.

Test by substituting 2 for x.

18. $\dfrac{1}{x^2-4}-\dfrac{1}{x-2}$.

PROCESS

$$\frac{1}{x^2-4}-\frac{1}{x-2}\equiv\frac{1}{x^2-4}-\frac{x+2}{x^2-4}\equiv\frac{1-(x+2)}{x^2-4}\equiv\frac{1-x-2}{x^2-4}$$
$$\equiv\frac{-x-1}{x^2-4}\equiv-\frac{x+1}{x^2-4}.$$

In this example, the fifth step gets rid of the negative numerator by changing the signs of the fraction. The fraction can be made positive as follows:

Changing all the signs of both terms of $\dfrac{-x-1}{x^2-4}$,

$$\frac{-x-1}{x^2-4}\equiv\frac{x+1}{-x^2+4}$$

and rearranging the terms,

$$\equiv\frac{1+x}{4-x^2}.\quad \textit{Ans.}$$

Test: Making $x \equiv 3$ in the original example,

$$\frac{1}{x^2-4}-\frac{1}{x-2}\equiv\frac{1}{9-4}-\frac{1}{3-2}\equiv\frac{1}{5}-1\equiv-\frac{4}{5}.$$

The answer $-\frac{x+1}{x^2-4}$ becomes $-\frac{3+1}{9-4}$, or $-\frac{4}{5}$.

The answer $\frac{1+x}{4-x^2}$ becomes $\frac{1+3}{4-9}=\frac{4}{5}$, or $-\frac{4}{5}$.

19. $\frac{1}{x^2-1}+\frac{1}{x-1}$.

20. $\frac{1}{x+1}-\frac{1}{x^2-1}$.

21. $\frac{1}{x-1}-\frac{1}{x^2-1}$.

22. $\frac{1}{x^2-1}-\frac{1}{x-1}$.

23. $\frac{1}{x^2-1}-\frac{1}{x+1}$.

24. $\frac{1}{x-2}-\frac{1}{x^2-4}$.

25. $\frac{1}{x^2-4}+\frac{1}{x-2}$.

26. $\frac{1}{x^2-4}-\frac{1}{x-2}$.

27. $\frac{1}{x^2-4}-\frac{1}{x+2}$.

28. $\frac{1}{x+2}-\frac{1}{x^2-4}$.

29. $\frac{x}{x^2-1}-\frac{x}{x-1}$.

30. $\frac{x}{x^2-1}-\frac{x}{x+1}$.

31. $\frac{x}{x^2-4}+\frac{x}{x+2}$.

32. $\frac{x}{x^2-4}-\frac{x}{x+2}$.

33. $\frac{x}{x+2}-\frac{x}{x^2-4}$.

SUPPLEMENTARY EXERCISES

34. $\frac{2}{x^2+5x+6}-\frac{3}{x+2}$.

PROCESS

Since $x^2+5x+6\equiv(x+2)(x+3)$, it is the common denominator.

$$\frac{2}{x^2+5x+6}-\frac{3}{x+2}\equiv\frac{2}{x^2+5x+6}-\frac{3(x+3)}{x^2+5x+6}\equiv\frac{2-3x-9}{x^2+5x+6}$$

$$=\frac{-3x-7}{x^2+5x+6}=-\frac{3x+7}{x^2+5x+6}. \quad \textit{Ans.}$$

Test: Substitute 1 for x.

35. $\frac{1}{x^2+5x+6}+\frac{1}{x+3}.$

36. $\frac{1}{x^2-5x+6}-\frac{1}{x-2}.$

37. $\frac{1}{x^2-5x+6}-\frac{1}{x-3}.$

38. $\frac{1}{x^2+5x+6}+\frac{1}{x+2}.$

39. $\frac{1}{x^2-5x+6}+\frac{1}{x-3}.$

40. $\frac{3}{x^2+5x+6}+\frac{x}{x+3}.$

41. $\frac{3}{x^2+5x+6}-\frac{x}{x+2}.$

42. $\frac{2}{x^2-5x+6}+\frac{x}{x-3}.$

43. $\frac{2}{x^2-5x+6}-\frac{x}{x-3}.$

44. $\frac{3}{x^2-5x+6}-\frac{x}{x-2}.$

45. $\frac{x-3}{x-2}-\frac{x-2}{x-3}.$

Process

The lowest common denominator is $(x-2)\,(x-3)$; or, x^2-5x+6.

$$\frac{x-3}{x-2}-\frac{x-2}{x-3}\equiv\frac{(x-3)\,(x-3)}{(x-2)\,(x-3)}-\frac{(x-2)\,(x-2)}{(x-3)\,(x-2)}$$

$$\equiv\frac{x^2-6x+9-(x^2-4x+4)}{x^2-5x+6}$$

$$\equiv\frac{x^2-6x+9-x^2+4x-4}{x^2-5x+6}\equiv\frac{-2x+5}{x^2-5x+6}$$

$$\equiv-\frac{2x-5}{x^2-5x+6}.\quad \textit{Ans.}$$

The L. C. D. of $x-2$ and $x-3$ is their product, $(x-2)(x-3)$.

As the denominator of the first fraction is $x-2$, this fraction is changed to one having the common denominator by multiplying its numerator and its denominator by $x-3$, the other factor of the L. C. D. In the same way the second fraction is changed by multiplying its numerator and its denominator by $x-2$, which operations are indicated in the first step.

In the second step, the indicated multiplications are performed, the new numerator of the second fraction being placed in a parenthesis, which is removed in the third step.

In the fourth step the numerator is simplified, and in the fifth the first term of the numerator of the fraction is made positive by changing the sign of the fraction.

Test: Substituting 4 for x, the original example becomes

$$\frac{x-3}{x-2}-\frac{x-2}{x-3}\equiv\frac{4-3}{4-2}-\frac{4-2}{4-3}\equiv\frac{1}{2}-\frac{2}{1}\equiv-\frac{3}{2}.$$

The answer becomes

$$-\frac{2x-5}{x^2-5x+6}=-\frac{8-5}{16-20+6}=-\frac{3}{2}.$$

46. $\frac{x+3}{x+2}+\frac{x+2}{x+3}$.

47. $\frac{x+3}{x+2}-\frac{x+2}{x+3}$.

48. $\frac{x-2}{x-3}+\frac{x-3}{x-2}$.

49. $\frac{x-2}{x-3}-\frac{x-3}{x-2}$.

50. $\frac{x-2}{x-3}+\frac{x+2}{x+3}$.

51. $\frac{x+1}{x+2}+\frac{x+2}{x+3}$.

52. $\frac{x-1}{x+2}+\frac{x-2}{x+3}$.

53. $\frac{x+1}{x+2}-\frac{x+2}{x+3}$.

54. $\frac{x-1}{x+2}-\frac{x-2}{x+3}$.

55. $\frac{x-2}{x-3}+\frac{x-2}{x+3}$.

56. $\frac{x+1}{x+2}-\frac{x+2}{x+4}$.

57. $\frac{x+3}{x+4}-\frac{x+4}{x+3}$.

58. $\frac{x+3}{x-4}-\frac{x+4}{x-3}$.

59. $\frac{x+1}{x+5}-\frac{x+1}{x-6}$.

60. $\frac{x-1}{x-5}-\frac{x-1}{x-6}$.

MULTIPLICATION AND DIVISION

To multiply fractions write the product of the numerators over the product of the denominators, first rejecting factors common to a numerator and a denominator.

To divide by a fraction, multiply by the fraction inverted.

SIGHT EXERCISES

Give results:

1. $\frac{5c}{a^2}\times\frac{a^2b^2}{10cx^2}$.

2. $\frac{5c}{a^2}\div\frac{10cx^2}{a^2b^2}$.

3. $\frac{3ab}{5x}\times\frac{5x^2}{3aby}$.

4. $\frac{6ab^2}{25x^2}\div\frac{3ab}{5x^2}$.

5. $\frac{3m^2n^3}{4xy}\times\frac{8x^2y^2}{9mn}$.

6. $\frac{3x}{4}\div\frac{11}{12}$.

7. $\frac{a^2}{xy}\div\frac{xy}{a^3}$.

8. $\frac{3x+1}{9}\times\frac{3}{4x}$.

9. $\frac{x+3}{x+2}\times\frac{x-3}{x-2}.$

10. $\frac{(x+2)(x+3)}{x+4}\times\frac{x+4}{x+2}.$

11. $\frac{x^2+5x+6}{x+4}\times\frac{x+4}{x+2}.$

12. $\frac{x^2-9}{x^2-4}\times\frac{x-2}{x-3}.$

13. $\frac{x^2-16}{x^2-25}\times\frac{x+5}{x+4}.$

14. $\frac{x^2-25}{x^2-9}\times\frac{x+3}{x-5}.$

15. $\frac{3x+1}{9}\div\frac{4x}{3}.$

WRITTEN EXERCISES

Perform the indicated operations:

1. $\frac{x^2+5x+6}{x^2-9x+20}\times\frac{x^2-7x+12}{x^2+x-6}\div\frac{x^2-x-6}{x^2+x-30}.$

Rewriting the foregoing with the factors indicated and inverting the third fraction the work takes the following form:

Process

$$\frac{(x+2)(x+3)}{(x-4)(x-5)}\times\frac{(x-3)(x-4)}{(x-2)(x+3)}\times\frac{(x+6)(x-5)}{(x+2)(x-3)}\equiv\frac{x+6}{x-2},\ \textit{Ans.}$$

the factors common to both a numerator and a denominator being rejected.

Test: Making $x\equiv 1$, the given expressions become

$$\frac{1+5+6}{1-9+20}\times\frac{1-7+12}{1+1-6}\div\frac{1-1-6}{1+1-30}$$

$$\equiv\frac{12}{12}\times\frac{6}{-4}\div\frac{-6}{-28}\equiv 1\times\left(-\frac{3}{2}\right)\times\left(\frac{14}{3}\right)\equiv -7.$$

The foregoing answer becomes $\frac{1+6}{1-2}\equiv\frac{7}{-1}\equiv -7.$

2. $\frac{x^2+5x+6}{x^2-9x+20}\times\frac{x^2-7x+12}{x^2+x-6}.$

3. $\frac{3a^2x}{4by}\times\frac{2b^2x}{9ay}\times\frac{3xy^2}{2ab^3}.$

4. $\frac{3a^2x}{4bc^3}\times\frac{2a^2b^2}{5cd^2x}\div\frac{3}{abc}.$

5. $\frac{3a^2b}{5x^2y^4}\times\frac{15x^3y}{8ab^4}\times\frac{10a}{9x}.$

6. $\frac{5a^2b}{7x^2y^4}\times\frac{8x^3y}{15ab^4}\times\frac{21bx}{16a}.$

7. $\frac{9x^2y}{10mn^2}\times\frac{5m^2n}{3xy^2}\div\frac{8mx}{9ny}.$

8. $\dfrac{3a^2x}{by} \times \dfrac{4b^2y^2}{9ax^2} \div 2ay.$

9. $\dfrac{5am^2}{21bn^2} \times \dfrac{7b^2n}{10a^2m} \div \dfrac{1}{ab}.$

10. $\dfrac{7x^3y}{12ab} \times \dfrac{8a^3}{21y^3} \div \dfrac{2a}{7xy^2}.$

11. $\dfrac{x+y}{(a-b)(a+b)} \times \dfrac{a+b}{(x-y)(x+y)}.$

12. $\dfrac{m+n}{x-y} \times \dfrac{x^2-y^2}{m^2-n^2}.$

13. $\dfrac{x^2-y^2}{x^2+y^2} \times \dfrac{4x}{x+y}.$

14. $\dfrac{x^2+y^2}{x^2-y^2} \div \dfrac{4x}{x+y}.$

15. $\dfrac{m^2-n^2}{m} \div \dfrac{m+n}{n}.$

16. $\dfrac{2x+3y}{x+y} \div \dfrac{x-y}{2x-3y}.$

17. $\dfrac{a}{a+x} \div \dfrac{a^2-a}{a+x}.$

18. $\dfrac{4m^2-9n^2}{x^2-y^2} \div \dfrac{2m-3n}{x+y}.$

19. $\dfrac{a^2}{a^2-b^2} \div \dfrac{2ab}{a^2-2ab+b^2}.$

SUPPLEMENTARY EXERCISES

20. $\left(1-\dfrac{2ab}{a^2+b^2}\right) \div \left(1-\dfrac{b}{a}\right).$

Process

Changing the dividend and the divisor to simple fractions,

$$1-\frac{2ab}{a^2+b^2} \equiv \frac{a^2+b^2}{a^2+b^2} - \frac{2ab}{a^2+b^2} \equiv \frac{a^2-2ab+b^2}{a^2+b^2}.$$

$$1-\frac{b}{a} = \frac{a}{a} - \frac{b}{a} = \frac{a-b}{a}.$$

Substituting the simple fractions, inverting the divisor, and canceling,

$$\left(1-\frac{2ab}{a^2+b^2}\right) \div \left(1-\frac{b}{a}\right) \equiv \frac{a^2-2ab+b^2}{a^2+b^2} \times \frac{a}{a-b}$$

$$\equiv \frac{a(a-b)}{a^2+b^2} \equiv \frac{a^2-ab}{a^2+b^2}. \quad \textit{Ans.}$$

Test by substituting 2 for a and 1 for b.

21. $\left(1+\frac{1}{x}\right)\div\left(1+\frac{1}{y}\right).$

22. $\left(x-\frac{y}{z}\right)\div\left(x+\frac{y}{z}\right).$

23. $\left(1+\frac{6}{x-6}\right)\div\left(2+\frac{3}{x-6}\right).$

24. $\left(\frac{x+y}{y}+\frac{y}{x+y}\right)\div\left(\frac{1}{x}+\frac{1}{y}\right).$

25. $1\div\left\{1-\left[1\div\left(1+\frac{1}{x}\right)\right]\right\}.$

26. $1\div\left\{a-\left[a\div\left(1-\frac{b^2}{a^2}\right)\right]\right\}.$

27. $a\div\left\{b-\left[c\div\left(a-\frac{1}{b}\right)\right]\right\}.$

28. $1\div\left\{x-\left[(x^2-1)\div\left(x+\frac{1}{x-1}\right)\right]\right\}$

29. $\left(\frac{a}{b}+\frac{a+b}{a-b}\right)\div\left(\frac{b}{a}+\frac{a-b}{a+b}\right).$

30. $\left(1-\frac{x-3y}{x+y}\right)\div\left(\frac{2x+y}{x-y}-2\right).$

31. $\left[1\div\left(1+\frac{1}{xy}\right)\right]\times\left[1\div\left(1-\frac{1}{xy}\right)\right].$

32. $\left(x-2-\frac{2}{x-5}\right)\div\left(x-3-\frac{1}{x-5}\right).$

33. $\left(\frac{a+3}{2}-\frac{2}{a+3}\right)\div\left(\frac{1}{2}-\frac{1}{a+3}\right).$

34. $(x-1)\div\left(2x-\frac{x+3}{x+1}\right).$

35. $\left(\frac{a^2-b^2-1}{2b}-1\right)\div\left(1+\frac{a^2-b^2+1}{2b}\right).$

36. $x^2-1-\left[(x-1)\div\left(1-\frac{x}{x+1}\right)\right].$

Simplify the following complex fractions:

37. $\dfrac{\dfrac{x^2y}{x^2-y^2}}{\dfrac{xy^2}{x+y}}$.

38. $\dfrac{\dfrac{a^3-x^3}{2ax+x^2}}{\dfrac{a-x}{x}}$.

39. $\dfrac{\dfrac{m^2n+nxy}{m^2y}}{\dfrac{m^2z+xyz}{my^2}}$.

40. $\dfrac{\dfrac{a^2-16}{a^2-9}}{\dfrac{(a-4)^2}{(a+3)^2}}$.

41. $\dfrac{\dfrac{x^2+xy+y^2}{x+y}}{\dfrac{x^2-xy+y^2}{x-y}}$.

42. $\dfrac{\dfrac{a^2+6a+8}{a^2-16}}{\dfrac{a^2-2a-8}{a-4}}$.

43. $\dfrac{\dfrac{m^2-7m+10}{m^2-3m+2}}{\dfrac{m^2-4m+4}{m^2-1}}$.

44. $\dfrac{\dfrac{a^2+3a-28}{a^2-4a-12}}{\dfrac{a^2+6a-40}{a^2-3a-18}}$.

45. $\dfrac{\dfrac{ab-2a+2b-4}{a-3}}{\dfrac{ab-2a-3b+6}{a-2}}$.

46. $\dfrac{\dfrac{xy-4x+3y-12}{y+2}}{\dfrac{xy-2x+3y-6}{y-4}}$.

47. $\dfrac{\dfrac{2mn-2m+3n-3}{2m-6}}{\dfrac{2mn-2m-n+1}{4m-2}}$.

48. $\dfrac{\dfrac{ax-5a+5x-25}{a^2-25}}{\dfrac{ax+5a-5x-25}{a^2-10a+25}}$.

Ratio and Proportion

RATIO

The ratio between two numbers of the same kind may be indicated by writing the numbers in the form of a fraction.

Thus, the ratio of 37 cows to 73 cows may be written $\frac{37}{73}$, the ratio of 6 days to 3 weeks may be written $\frac{6}{21}$, the 3 weeks being changed to 21 days in order that the numbers may be of the same kind, and is then reduced to $\frac{2}{7}$.

While the numbers to be compared may be concrete, the ratio between them is always abstract.

The first term of a ratio is called the **antecedent** and the second the **consequent**, the two terms constituting a **couplet**.

The antecedent of the couplet is the numerator of the fraction employed to denote a ratio, and the consequent is the denominator. The consequent should be expressed, even when it is unity.

Thus, the ratio of 16 to 2 is $\frac{16}{2}$ or 8 : 1.

A ratio is expressed, as a rule, in whole numbers and in lowest integral terms. While the ratio of 100 to 150 might be expressed as 1 : $1\frac{1}{2}$, 4 : 6, etc., it is generally given as 2 : 3.

In "stating" a problem, unknown numbers having a given ratio are generally expressed in the terms of the ratio. Thus, two unknown numbers bearing the ratio of 7 to 9 are represented by $7x$ and $9x$, respectively.

SIGHT PROBLEMS

1. Two numbers have the ratio of 3 to 5. When the greater is 120, what is the other?

2. Give two multiples of x that have the ratio of 4 to 11.

3. What two numbers in the ratio of 5 to 6 have 99 for their sum?

4. Two numbers in the ratio of 5 to 3 have a difference of 60; find the numbers.

5. The product of two numbers is 96 and their ratio is 3 to 2. What are the numbers?

PROPORTION

Since proportion is the expression of equality between two ratios, the missing term of a proportion may be found by solving an equation whose respective members are the two ratios expressed as fractions.

Thus, the value of x in

$$3 : 4 :: x : 12$$

may be found by writing the proportion in the form of an equation as follows:

$$\frac{3}{4} = \frac{x}{12}.$$

SIGHT EXERCISES

Give the missing term:

1. $\frac{3}{4}=\frac{9}{x}$.
2. $\frac{x}{5}=\frac{20}{25}$.
3. $\frac{6}{x}=\frac{7}{14}$.
4. $\frac{8}{11}=\frac{x}{44}$.
5. $\frac{9}{17}=\frac{27}{x}$.
6. $\frac{x}{13}=\frac{4}{52}$.
7. $\frac{7}{x}=\frac{21}{9}$.
8. $\frac{x}{7}=\frac{36}{14}$.
9. $\frac{3}{8}=\frac{12}{x}$.
10. $3:4::9:x$.
11. $6:x::7:14$.
12. $9:17::27:x$.
13. $x:13::4:52$.
14. $7:x::21:9$.
15. $3:8::12:x$.

In any proportion the first and fourth terms are called the **extremes**, the second and third are called the **means**.

Since $8:17::24:x$ is the same as $\frac{8}{17}=\frac{24}{x}$, which can be solved by "clearing of fractions," thus:

$$8(x)=24(17)$$

the method of solving an example in proportion may be formulated as follows:

To solve a problem in proportion form an equation whose respective members will be the product of the means and the product of the extremes.

WRITTEN EXERCISES

Find the value of x:

1. $\frac{x-4}{x+4}=\frac{2}{3}$.
2. $\frac{x+5}{3}=\frac{x-5}{2}$.
3. $\frac{3}{2}=\frac{x-6}{x+6}$.
4. $\frac{x+2}{x+6}=\frac{x+10}{x+18}$.
5. $\frac{x^2-9}{x^2-1}=\frac{x^2-13}{x^2-7}$.
6. $\frac{x^2-12}{x^2+1}=\frac{x^2-10}{x^2+5}$.

Find one value of x:

7. $x^2 - x + 1 : 1 :: 9 : x + 1$.

8. $x^2 + 9 : x^2 - 9 :: 17 : 8$.

9. $x^2 + 9 : x^2 - 8 :: x^2 - 9 : x^2 - 17$.

10. $\frac{63}{x} + x : \frac{63}{x} - x :: 8 : 1$.

WRITTEN PROBLEMS

1. By what number must each term of the ratio of 14 to 19 be increased to make it equal to the ratio of 4 to 5?

$$14 + x : 19 + x :: 4 : 5$$

2. Two numbers have the ratio of 4 to 5. When the smaller is 24, what number subtracted from each will make the ratio of the respective remainders 7 : 9?

3. A boy's age is to his father's age as 1 to 8. In 18 years the ratio will be 1 to 2. What are their respective ages?

4. The sum of the squares of two consecutive odd numbers is to the square of their sum as 37 is to 72. Find the numbers.

Take $x - 1$ and $x + 1$.

5. The difference between two numbers is 10, and their product is to the sum of their squares as 36 is to 97. What are the numbers?

Let $x + 5$ and $x - 5$ represent the numbers.

ARRANGEMENT OF TERMS

Since in a proportion the product of the extremes is equal to the product of the means, the terms of a proportion may be arranged in various ways; as follows:

(*a*) $6 : 9 :: 8 : 12$
(*b*) $6 : 8 :: 9 : 12$
(*c*) $8 : 6 :: 12 : 9$
(*d*) $8 : 12 :: 6 : 9$
(*e*) $9 : 6 :: 12 : 8$
(*f*) $9 : 12 :: 6 : 8$
(*g*) $12 : 8 :: 9 : 6$
(*h*) $12 : 9 :: 8 : 6$

Review Exercises and Problems

SIGHT EXERCISES

1. Give the factors of $a^4 - b^4$, $a^4 - 16$, $a^3 - 8$, $a^3 + 8$.

2. Give the product of $a^2 - 3a + 9$ by $a + 3$.

3. Give the quotient of $a^3 - 64$ by $a - 4$. Of $a^6 - 64$ by $a^2 - 4$. Of $a^6 - 64$ by $a^3 + 8$.

4. Give two factors of $a^6 - 64$. Of $4a^4 - 1$.

5. Give the highest common factor of $x^4 - 16$ and $x^2 - 4$. Of $x^2 + 4x + 4$ and $x^2 - 4$.

6. Give the factors of the L. C. M. of $x^2 - 4$ and $x^2 + 4x + 4$.

7. Give the square of $2a + 3b$.

8. Give the square root of $9a^2 + 24ab + 16b^2$.

9. Give the value of each of the following: $(100 + 1)^2$, $(100 - 1)^2$, $101^2 - 99^2$.

10. Give the factors of $(a + b)^2 - 1$. Of $(a^2 + 2ab + b^2) - 1$.

SIGHT PROBLEMS

1. A, who can work twice as fast as B, requires 6 days to dig a ditch. In how many days can A and B dig it working together?

2. (*a*) The product of two consecutive numbers is 72; what are the numbers? Give the square root (*b*) of $72\frac{1}{4}$. (*c*) Of 7225.

3. Give the cost of

(*a*) 73 lb. @ 77¢
(*b*) 64 yd. @ 66¢
(*c*) 48 cows @ $42
(*d*) 94 bu. @ 86¢

4. Two boys had the same number of marbles. After one had lost 18 and the other had bought 10, the former had only one half as many as the latter. How many had each at first?

5. A man had originally the same number of ducks as hens. After exchanging 15 ducks for 25 hens he had three times as many hens as he had ducks. How many of each did he then have?

6. When three times a number is taken from 60, the remainder is 20 more than twice the number. Find the number.

7. A bought 90 pounds of sugar and B bought 70 pounds. In how many weeks will they have the same quantity left if B uses 2 pounds per week and A uses 3 pounds per week?

8. Two boys start at the same time from the opposite ends of a field 440 yards wide, and meet in 4 minutes. If one goes 10 yards faster per minute than the other, how far does each travel?

9. A man received for a number of calves as many dollars apiece as there were calves; if he had sold one more calf and received for each $1 more per head, his receipts would have been $21 larger. How many did he sell?

10. By receiving for a number of horses as many dollars each as there were horses, a man received a certain sum. How much more or less would he have received if he had sold 5 horses fewer and received $5 more apiece for them?

WRITTEN EXERCISES

Find the value of x.

1. $5x^2 + 2x - 100 = (5x + 9)(x - 3) - 9.$
2. $7(x + 2) + 8(x + 3) = 13(x + 4).$
3. $4x - \{7x - 2[15 - 5x - (3 - 7x)]\} = 30.$
4. $(x - 3)(x + 4) = (x + 5)(x - 6) + 36.$
5. $(2x + 3)(4x - 7) = 8(x - 1)^2 + 41.$
6. $\frac{x+1}{3} - \frac{x-1}{4} + 4x = 12 + \frac{2x-1}{6}.$

7. $x+\frac{8}{3}+\frac{7-4x}{4}+\frac{4x-7}{8}=2x+\frac{13}{24}.$

8. $\frac{1}{2}(x+6)-\frac{1}{3}x=\frac{1}{6}(8-x)+\frac{1}{8}(3x-5).$

9. $\frac{7x}{3}-\frac{11}{6}(x+3)=\frac{3x-1}{5}-6-x-\frac{1-3x}{2}.$

10. $\frac{1-x}{2}-\frac{2-x}{3}+\frac{3-x}{4}-\frac{4-x}{5}+\frac{5-x}{6}=\frac{17}{40}.$

WRITTEN PROBLEMS

1. The difference between two numbers is 10 times the smaller number, and 6 times the smaller subtracted from twice the larger gives a remainder of 80. Find the numbers.

2. A boy sold apples, peaches, and pears at 2 for 3 cents, 2 for 5 cents, and 3 for 10 cents, respectively. How many did he sell of each if he sold twice as many peaches as apples, as many pears as apples and peaches together, receiving 99 cents for those he sold?

3. A man bought a certain number of marbles at the rate of 3 for 2 cents, twice as many at the rate of 5 for 4 cents, and three times as many at the rate of 10 for 3 cents, paying 95 cents in all. How many of each did he buy?

4. Two trains leaving the same point at the same time and going in opposite directions are 98 miles apart in two hours. How many miles does each go in an hour if one requires as long to go 3 miles as the other requires to go 4 miles?

5. Two trains traveling in the same direction leave the same place at the same time, one going $7\frac{1}{2}$ miles while the other goes $5\frac{1}{2}$. How many miles per hour does each travel if the distance between them in 4 hours is 260 miles?

6. A tank is fed by two pipes, one of which requires 20 minutes to fill it, and the second 30 minutes. When the tank is empty, how long will it take both to fill it if the water is escaping through a pipe that requires an hour to empty the tank when it is full?

7. A grocer mixes a quantity of coffee costing 20 cents a pound with a quantity costing 25 cents a pound, making a mixture containing 50 pounds costing 23 cents a pound. How many pounds of each does he take?

8. If a cubic foot of gold weighs 19,250 ounces avoirdupois, and a cubic foot of silver weighs 10,500 ounces avoirdupois, what part of a cubic foot of gold is contained in a cubic foot of an alloy of gold and silver weighing 16,500 ounces?

Let $x =$ fraction of gold, $1 - x =$ fraction of silver.

9. How many pounds of 25-cent coffee must be mixed with 30 pounds of 19-cent coffee to make a mixture costing 21 cents per pound?

10. If a piece of gold of a certain bulk weighs 77 ounces, a piece of silver of the same bulk weighs 42 ounces, and a piece of a mixture of gold and silver of the same bulk weighs 66 ounces, what fraction of the latter is gold and what fraction is silver?

CHAPTER IV

LINEAR EQUATIONS

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ONE UNKNOWN NUMBER

A **linear equation** is an equation that contains only the first power of the unknown number.

Linear equations are also called **equations of the first degree** or **simple equations.**

SOLUTION OF LINEAR EQUATIONS

The solution of an equation is the operation of finding such a value for the unknown number as will make both members of the equation identical when this value is substituted for the unknown number.

Thus, in solving the equation $3x - 8 = 2x - 5$, the value 3 obtained for x is tested.

Substituting 3 for x, the first member becomes $3 \cdot 3 - 8 \equiv 9 - 8 \equiv 1$; and the second becomes $2 \cdot 3 - 5 \equiv 6 - 5 \equiv 1$.

As 3 makes both members identical, it is the required value.

The value of an unknown number in a given equation is called the **root** of the equation.

A linear equation containing one unknown number has but one root.

TRANSFORMATIONS OF EQUATIONS

During the process of solving an equation, several transformations may be necessary. Thus, to find the value of x in

$$x - \frac{x-2}{4} = \frac{5x+6}{3} - x, \qquad (a)$$

it is convenient to transform the given equation into one containing only entire terms.

This is accomplished by multiplying both members of the original equation by a common multiple of the denominators of the fractional terms. Multiplying both terms of (*a*) by 12, the L. C. M. of 4 and 3, the transformed equation (*b*) is obtained.

The successive steps of the operation may be shown as follows, although in practice, (*b*) is obtained directly from (*a*) by writing at once the product of each term by 12.

$$12\left(x - \frac{x-2}{4}\right) = 12\left(\frac{5x+6}{3} - x\right)$$

or, $$12x - 3(x-2) = 4(5x+6) - 12x$$

or, $$12x - (3x-6) = 20x + 24 - 12x$$

or, $$12x - 3x + 6 = 20x + 24 - 12x \qquad (b)$$

The process of transforming an equation containing fractions to one composed exclusively of entire numbers, by multiplying both members by the same number is called, for convenience, **clearing of fractions.**

Another transformation produces an equation which comprises in one member, generally the first, all the unknown numbers and in the other all the known numbers. As the first member of (*b*) contains the known number 6, this is subtracted from both members, making the new equation,

$$12x - 3x(+6-6) = 20x + 24 - 12x - 6.$$

In the same way $20x$ is subtracted from both members and $12x$ is added to both:

$$12x - 3x(+6-6) - 20x + 12x = (20x - 20x) + 24(-12x + 12x) - 6.$$

In practice, however, the terms inclosed in parentheses are rejected, as they are identical, but with opposite signs, which makes their combined value zero. The transformed equation takes this form, which is written directly from (*b*)

$$12x - 3x - 20x + 12x = 24 - 6. \qquad (c)$$

By comparing (*b*) with (*c*) it will be observed that $+6$ in the first member of the former is replaced by -6 in the second member of the latter, $20x$ of the second member of (*b*) by $-20x$ in the first member of (*c*), and $-12x$ in the second member of (*b*) by $+12x$ in the first member of (*c*).

The change of (*b*) into (*c*) by increasing and diminishing both members of an equation by the same numbers is for convenience called **transposition**, the method in practice being to write in the other term with the opposite sign any number omitted from either term.

A third transformation consists in simplifying both members, (*c*) now becoming,

$$x = 18. \qquad (d)$$

This root is tested by substituting it for x in each member of the original equation.

Transformations are not necessarily made in any given order. Some pupils combine like entire quantities at first, and also transpose negative fractions to make them positive, (*a*) then becoming

$$2x = \frac{5x+6}{3} + \frac{x-2}{4}$$

which is then transformed into

$$24x = 20x + 24 + 3x - 6$$

and finally into

$$x = 18.$$

Frequently the preceding transformations produce an equation in which the unknown number contains a coefficient; as $5x = 30$, which is changed into $x = 6$ by dividing both members by 5. An equation in the form of $-4x = 28$, is transformed into $x = -7$, by dividing both members by -4. The equation $-x = -3$ may be transformed into $x = 3$ by multiplying both members by -1, or into $3 = x$ by transposition.

In transposing a fraction, only the sign of the fraction is changed, and not the sign connecting the terms of the numerator or of the denominator. In transposing a polynomial contained in a parenthesis, only the sign preceding the parenthesis is changed.

The authority for the foregoing changes made in the members of an equation during the process of solving it may be summed up as follows:

The equality of an equation is not affected when both members are

(1) increased, (2) diminished, (3) multiplied, or (4) divided by the same number.

The foregoing is generally expressed in four statements which are called *axioms*. They are considered self-evident; but they are subject to some explanations and limitations, which will be given later.

WRITTEN EXERCISES

Find the value of x:

1. $\frac{7x-8}{11}+\frac{3x-3}{2}=2x-\frac{31-x}{22}$.

2. $9-\frac{76-4x}{3}=\frac{2x+1}{9}-\frac{75-3x}{18}$.

3. $2x+\frac{3x-5}{2}=17-\frac{2x-4}{3}$.

4. $\frac{46x+4}{3}-\frac{8x+4}{5}=20x-\frac{7x+10}{2}$.

5. $\frac{2x-2}{3}-\frac{x-1}{2}=3x-\frac{5x-7}{4}-13$.

6. $x-\frac{3x+17}{5}=\frac{20-x}{2}-\frac{6x-8}{7}-\frac{4x-4}{5}$.

7. $\frac{x+5}{2}-\frac{x-3}{6}+\frac{27-x}{3}=x-10+\frac{25-x}{4}$.

8. $\frac{2x}{3}+\frac{5+x}{2}-\frac{3x-2}{5}=x+\frac{x-2}{3}-\frac{17}{12}$.

9. $\frac{4x+7}{7}+\frac{7x+1}{5}-\frac{13x}{2}=3-\frac{4+9x}{2}$.

10. $x(x+5)-(x-3)(x-4)-7(x+3)+8=0.$

Solution

Given $x(x+5)-(x-3)(x-4)-7(x+3)+8=0.$

Performing the indicated multiplications,

$$x^2+5x-(x^2-7x+12)-(7x+21)+8=0.$$

Removing parentheses,

$$x^2+5x-x^2+7x-12-7x-21+8=0.$$

Combining and transposing,

$$5x=25,$$

$$\therefore x=5. \quad \textit{Ans.}$$

Test : Substituting 5 for x, the first member becomes,

$$\begin{aligned}&5(5+5)-(5-3)(5-4)-7(5+3)+8\\ &\equiv 5\cdot 10-2\cdot 1-7\cdot 8+8\\ &\equiv 50-2-56+8\\ &\equiv 58-58\equiv 0.\end{aligned}$$

11. $2(x-1)(x+1)=x(2x+3).$

12. $(x+1)(x+2)=(x-2)(x+4).$

13. $(x+1)^2+(x+2)^2+(x+3)^2=3x^2.$

14. $(x+7)(4x-10)=(2x+6)(2x-1).$

15. $(3x-5)(2+x)=(x+1)(3x-4).$

16. $(8x+1)(2-x)=(2x+1)(7-4x).$

17. $8+(x-4)(2+x)=(x-3)(7+x).$

18. $(x-6)(6+x)-(x-7)(7+x)=x.$

19. $x(x+6)-(x-2)(x-3)=8x-1.$

20. $4-9x=(3x-3)^2+(4x+3)^2-(5x+2)^2.$

The following equations are cleared of fractions by cross multiplication:

21. $$\frac{5x-3}{87}=\frac{5x+7}{97}.$$

Multiplying both members by $97\cdot 87$,

$$97(5x-3)=87(5x+7).$$

22. $\frac{3x-1}{4}=\frac{9x-8}{11}$.

23. $\frac{7x+1}{18}=\frac{3x+5}{10}$.

24. $\frac{x+9}{17}=\frac{x-11}{7}$.

25. $\frac{2x+8}{7}=\frac{6x-2}{20}$.

26. $\frac{10x+3}{11}=\frac{7x-6}{5}$.

27. $\frac{4x+3}{13}=\frac{3x-12}{5}$.

28. $\frac{7x+1}{10}=\frac{8x-1}{11}$.

29. $\frac{4x+7}{23}=\frac{3x-1}{11}$.

30. $\frac{6x+3}{25}=\frac{4x+3}{17}$.

31. $\frac{5x+1}{3}=\frac{7x-2}{4}$.

SUPPLEMENTARY EXERCISES

32. $\frac{2x-5}{x+7}=\frac{2x-9}{x+3}$.

In solving an equation each member of which consists of a fraction, it is unnecessary to obtain the common denominator, especially if the denominators are prime to each other.

In this example, the L. C. D. is $(x+7)(x+3)$, and the respective fractions are equal to $\frac{(2x-5)(x+3)}{(x+7)(x+3)}$ and $\frac{(2x-9)(x+7)}{(x+7)(x+3)}$.

Since the denominators are equal, the numerators are equal, and the equation becomes

$$(2x-5)(x+3)=(2x-9)(x+7);$$

that is, the product of the first numerator by the second denominator is equal to the product of the second numerator by the first denominator.

33. $\frac{2x+4}{2x+1}=\frac{5x+1}{5x-5}$.

34. $\frac{x+4}{x+5}=\frac{4x+5}{4x+8}$.

35. $\frac{x+3}{x+4}=\frac{4x+1}{4x+4}$.

36. $\frac{3x-3}{x}=\frac{3x-1}{x+1}$.

37. $\frac{x-1}{x+1}=\frac{x-6}{x+4}$.

38. $\frac{2x-5}{2x-9}=\frac{x+7}{x+3}$.

39. $\frac{2x+4}{5x+1}=\frac{2x+1}{5x-5}$.

40. $\frac{x-2}{x-1}=\frac{2x+7}{2x+10}$.

41. $\frac{x-6}{x-5}=\frac{2x-1}{2x+2}$.

42. $\frac{5}{x+2}=\frac{3}{x+1}$.

43. $$\frac{5}{1-x}=\frac{7}{1+x}+\frac{6}{1-x^2}.$$

L. C. D. is $1-x^2$. Reducing the fractions in the original equations to fractions whose denominator is $1-x^2$, the given equation becomes $\frac{5(1+x)}{1-x^2}=\frac{7(1-x)}{1-x^2}+\frac{6}{1-x^2}$. Omitting the denominators, etc., it becomes $5+5x=7-7x+6$, etc.

44. $$\frac{6}{1+x}+\frac{x+1}{x-1}=\frac{3-x^2}{1-x^2}.$$

Solution

Given $$\frac{6}{1+x}+\frac{x+1}{x-1}=\frac{3-x^2}{1-x^2}.$$

Changing the sign of the denominator of the second fraction,

$$\frac{6}{1+x}-\frac{1+x}{1-x}=\frac{3-x^2}{1-x^2}.$$

Multiplying both members by $1-x^2$, the L. C. D.,

$$6(1-x)-(1+x)^2=3-x^2,$$

that is, $$6-6x-1-2x-x^2=3-x^2.$$

Combining and transposing,

$$-8x=-2.$$

$$\therefore x=\tfrac{1}{4}.\quad \textit{Ans.}$$

Test: When $x\equiv\frac{1}{4}$, the first member becomes

$$\frac{6}{1+\frac{1}{4}}+\frac{\frac{1}{4}+1}{\frac{1}{4}-1}\equiv\frac{24}{5}-\frac{5}{3}\equiv\frac{72-25}{15}\equiv\frac{47}{15}.$$

The second member becomes

$$\frac{3-\frac{1}{16}}{1-\frac{1}{16}}\equiv\frac{48-1}{16-1}\equiv\frac{47}{15}.$$

45. $$\frac{12}{x-1}-\frac{8x}{x^2-1}=\frac{6}{x+1}.$$

46. $$\frac{4}{x-1}-\frac{3}{1+x}=\frac{15}{x^2-1}.$$

47. $$\frac{80}{x^2-4}=\frac{x+2}{x-2}-\frac{x-2}{x+2}.$$

48. $$\frac{1-x}{x+1}=\frac{3x-5}{1-x^2}+\frac{x+3}{1-x}.$$

49. $$\frac{2-x}{x+2}-2\left(\frac{x^2+x}{4-x^2}\right)=\frac{x+2}{x-2}.$$

50. $\dfrac{2x-1}{1+2x}=\dfrac{2x+1}{2x-1}-\dfrac{3x^2}{4x^2-1}$.

51. $\dfrac{2x+1}{2x-1}-\dfrac{2x-1}{2x+1}=\dfrac{8}{4x^2-1}$.

52. $\dfrac{1}{x^2-5x+6}=\dfrac{1}{x^2-13x+42}$.

53. $\dfrac{-3}{x^2-x-6}=\dfrac{-3}{x^2-13x+42}$.

54. $\dfrac{4x+3}{9}+\dfrac{7x-29}{5x-12}=\dfrac{8x+19}{18}$.

SOLUTION

Given $$\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}.$$

Transposing to the second member the fractions having a monomial denominator,

$$\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{4x+3}{9}.$$

Reducing the fractions in the second member to the same denominator,

$$\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{8x+6}{18}.$$

Combining the terms of the second member,

$$\frac{7x-29}{5x-12}=\frac{13}{18}.$$

Clearing of fractions,

$$18(7x-29)=13(5x-12).$$

That is, $$126x-522=65x-156.$$

Transposing and combining,

$$61x=366.$$

$$\therefore x=6. \quad \textit{Ans.}$$

Test :

Substituting 6 for x, the first member becomes

$$\frac{24+3}{9}+\frac{42-29}{30-12}=\frac{27}{9}+\frac{13}{18}=\frac{54+13}{18}=\frac{67}{18}.$$

The second member becomes $$\frac{48+19}{18}=\frac{67}{18}.$$

Equations of this type are solved by placing in one member the fraction having a polynomial denominator, and those having numerical denominators in the other, and first simplifying this member.

55. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}.$

Transpose $\frac{x}{4}$.

56. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

Transpose $-\frac{12x+2}{13x-16}$ and $\frac{5x-4}{9}$.

57. $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}.$

58. $\frac{2x-5}{5}+\frac{x-3}{2x-15}=\frac{4x-3}{10}-\frac{11}{10}.$

59. $x-\frac{3x-7}{9}=\frac{2x+4}{3}-\frac{7x-8}{3(2x+4)}.$

60. $\frac{4x+17}{18}-\frac{13x-2}{17x-32}=\frac{7x}{12}-\frac{13x+16}{36}.$

61. $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}.$

62. $11x+\frac{3x-5}{2}=12-\frac{2x-4}{3}+10x.$

63. $\frac{3x+8}{2}+5=3x+\frac{5x-4}{3}+\frac{8x+4}{5}.$

64. $\frac{x}{4}+3-\frac{2x-2}{3x-42}=\frac{7x-40}{12}-\frac{x-4}{3}.$

65. $\frac{x-9}{5}+1\frac{3}{20}=\frac{x}{5}+\frac{7}{10}-\frac{3(5x+3)}{8(2x-1)}.$

66. $\frac{9x+12}{21}-\frac{4x-10}{7}=\frac{7x-2}{6x-1}+\frac{7-x}{7}.$

67. $\frac{13x+54}{6}-\frac{5x+15}{x-9}=2x+\frac{x}{6}.$

STATEMENT OF PROBLEMS

Pupils able to solve an equation readily, frequently find difficulty in "stating" a problem.

The most frequent mistake is to "Let $x =$" the number called for in the problem, the area, for instance, in the following:

The perimeter of a rectangle is 62 rods and one side measures 14 rods less than the adjacent side; find the area.

In solving this problem the pupil's procedure should take the form indicated by the following questions and answers.

"What is required in this problem?"

"The area."

"What is required to ascertain the area?"

"The dimensions."

"Are the dimensions given?"

"Not specifically."

"Are any data given by which they can be ascertained?"

"Yes, the perimeter is given and the relation between the adjacent sides."

When the pupil has determined that the unknown numbers are the dimensions, and that they can be calculated, he proceeds as follows:

Let $x =$ shorter side.
Then $x + 14 =$ adjacent side,
and $x + (x + 14) + x + (x + 14) =$ perimeter $= 62$.

After the lengths of the adjacent sides are found, the area is obtained arithmetically.

WRITTEN PROBLEMS

1. Solve the foregoing.

2. The perimeter of a rectangle is 68 rods, and the base is 14 rods longer than the altitude. Find the diagonal.

NOTE. First obtain the dimensions.

3. Find the area of a right triangle whose perpendicular is 8 yards and whose hypotenuse is 2 yards longer than the base.

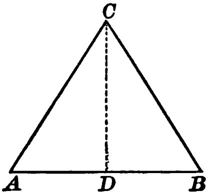
Let $x =$ base ; then $x+2 =$ hypotenuse.

$$(x+2)^2-(x)^2=(8)^2.$$

4. Find the diagonal of a rectangle whose dimensions are in the ratio of 5 to 12, and whose perimeter is 136 feet.

Represent the dimensions by $5x$ and $12x$.

5. The perimeter of an isosceles triangle is 36 yards and the base is 3 yards shorter than either of the equal sides. Find the area.



First find the length of the base and of the equal sides. Then calculate arithmetically the altitude of the triangle, taking one half the base as a side of a right triangle, of which one of the equal sides of the isosceles triangle forms the hypotenuse. Finally, calculate arithmetically the area of the triangle.

6. The difference between two numbers is $37\frac{1}{2}$ and the smaller is one sixth of the larger. Find their sum.

7. The sum of two numbers, one of which is $7\frac{1}{2}$ times the other, is 68. What is their difference?

8. The sum of one half and one third of a number is 30. Find the product of one fourth of the number by its one fifth.

9. After spending one third of his money and then one fourth of the remainder, a boy has 30 cents left. How much did he spend?

10. A man spent one fourth of his money, then two fifths of the remainder, then five sixths of the remainder, and had \$30 left. How much money did he spend?

If x represents the original money, the first remainder is $\frac{3x}{4}$; the second remainder is $\frac{3}{5}$ of $\frac{3x}{4}$; and the third remainder is $\frac{1}{6}$ of $\frac{3}{5}$ of $\frac{3x}{4}$; or $\frac{3x}{40}$, which is equal to 30. Number of dollars spent $= x - 30$.

11. After receiving a discount of 25% of the catalogue price, and 20% of the remainder, and 10% of what then

remained, a person paid $270 for a piano. What was the catalogue price? What was the total discount?

Representing the catalogue price by x, the first remainder is $\frac{3}{4}$ of x, the second is $\frac{4}{5}$ of $\frac{3}{4}$ of x, the third is $\frac{9}{10}$ of $\frac{4}{5}$ of $\frac{3}{4}$ of x. From the catalogue price deduct $270 to obtain the total discount.

12. If a certain principal amounts in 1 year 3 months at 6% simple interest to $172, what will be the amount of the same principal in 2 years 6 months?

Representing principal by x, the interest for 1 year will be $x \cdot \frac{6}{100}$, and for $1\frac{1}{4}$ years it will be $x \cdot \frac{6}{100} \cdot \frac{5}{4}$; or $\frac{3x}{40}$. The amount will be $x + \frac{3x}{40}$. Solve the equation $x + \frac{3x}{40} = 172$ to find the principal. Calculate the interest of the principal thus found for 2 years 6 months, and add it to the principal.

While the *interest* for 2 years 6 months is twice the interest for 1 year 3 months, the *amounts* for these periods do not bear the same ratio.

13. If the amount of a certain principal for 1 year 6 months is $327 when the rate is 6%, what will be the amount for the same time when the rate is 3%?

14. If the proceeds of a 60-day note discounted at 6% are $4752, how much could a man save by having it discounted at 5%?

15. Two trains start at the same time from points 280 miles apart and travel toward each other. How far will each travel to the meeting place if their rates of travel are in the ratio of 3 to 4?

16. If a train starts from A at 1 P.M. and a train running 10 miles less per hour starts at 2 P.M. from B, 320 miles from A, what are their respective rates of speed if they meet at 6 P.M.?

The distance (320 miles) is the sum of the distances made by one train in 5 hours and by the other in 4 hours.

17. If a train starts from M at 2 P.M., and a train running 10 miles faster per hour starts from N at 4 P.M., and they meet at 10 P.M., what is the rate of speed of each train, the distance from M to N being 410 miles?

18. A boy going 1 mile faster per hour overtakes in $1\frac{1}{2}$ hours another boy who has had $\frac{1}{2}$ hour start. How many minutes does each take to go a mile?

19. When 176 is divided into two parts having the ratio of $4\frac{1}{2}$ to $3\frac{1}{2}$, what is the difference between the parts?

20. The difference between two numbers having the ratio of $3\frac{3}{4}$ to $4\frac{1}{4}$ is 26. Find the sum of the numbers.

21. A merchant buys cloth at $2.50 per yard. At what price per yard should he sell it in order that his profit on 28 yards will be the selling price of 3 yards?

22. A merchant sells goods at an advance of 10%, the profit being $15 less than 10% of the selling price. Find the cost of the goods.

23. A man and his two sons can do a piece of work in 5 days; one of the sons can do $\frac{1}{2}$ as much as his father and the other can do $\frac{3}{10}$ as much as his father. How long would it take each to do the work by himself?

SOLUTION

Let $x =$ number of days the father requires.

Then $2x =$ number of days one son requires,

and $\frac{10x}{3} =$ number of days other son requires.

Therefore $\frac{1}{x} =$ part of work done by father in a day.

$\frac{1}{2x} =$ part of work done by one son in a day,

and $\frac{3}{10x} =$ part of work done by other son in a day.

By the terms of the problem

$\frac{1}{5} =$ the part done by all three in a day.

24. A man draws from the bank four fifths of his money. Later, he deposits a sum equal to two thirds of the sum remaining in the bank. If he then has $60 in the bank, how much had he originally?

25. A sum of money was divided among four persons, the first one's share being $\frac{1}{5}$ of the whole, the second one's being $\frac{4}{9}$ of the remainder, the third one's being $\frac{2}{5}$ of the sum then left, and the fourth one's being the balance, which was $2400. What was the share of each of the other three?

26. The owner of a three-story house receives $324 in rents per year. The rent of the first floor is $1\frac{1}{9}$ times that of the second floor, and the rent of the second is $1\frac{1}{8}$ times that of the third. What rent is received from each?

27. A buys $\frac{2}{3}$ of $\frac{3}{4}$ of a piece of cloth; B buys $\frac{2}{5}$ of $\frac{7}{8}$ of the same piece. What is the length of the piece, if A pays $40.50 more than B, the price of the cloth being $2.25 per yard?

A buys $\frac{x}{2}$ yards, B buys $\frac{7x}{20}$ yards.

$$\frac{x}{2}-\frac{7x}{20} = \text{Difference in number of yards bought.}$$

Difference in number of yards bought: $40.50 ÷ $2.25.

28. A boy on a trip went 14 miles on foot, then three and one half times as far by water, and the remainder by train. What was the length of his journey, if the distance traveled by train was three fourths of the trip?

29. A train starts from A at 8 A.M. going to B, 130 miles distant. At the same time a train traveling 5 miles less per hour starts from B, and meets the first at 10 A.M. At what distance from A do they meet?

30. A train starting from M at 9 A.M. travels toward N, 130 miles distant. At 8.10 A.M. a train leaves N, going 10 miles less per hour, and meets the other at 10:30 A.M. How many miles does each travel?

Simultaneous Linear Equations

TWO UNKNOWN NUMBERS

In the equation $x + 3y = 12$, x and y may have an indefinite number of values, as is shown in the following sets:

(*a*) $y = 1, x = 9.$	(*f*) $y = -1, x = 15.$
(*b*) $y = 2, x = 6.$	(*g*) $y = -2, x = 18.$
(*c*) $y = 3, x = 3.$	(*h*) $y = -3, x = 21.$
(*d*) $y = 4, x = 0.$	(*i*) $y = -4, x = 24.$
(*e*) $y = 5, x = -3.$	(*j*) $y = -5, x = 27.$

A single equation containing more than one unknown number is called an **indeterminate equation.**

When all of two or more independent linear equations can be satisfied by only one value of each unknown number, the equations are said to be **simultaneous.**

Thus in the linear equations:

$$x + 3y = 12, \quad x + y = 10$$

the only value of x that will satisfy both equations is $x = 9$, the corresponding value of y being 1.

The two equations

$$x + y = 6 \text{ and } 2x + 2y = 12$$

are not simultaneous equations, inasmuch as they are not independent, the latter equation being reducible to the former by dividing both members by 2.

Elimination

Two simultaneous equations are solved by combining them into a single equation containing but one unknown number, the second unknown number being **eliminated** from the new equation.

There are three general methods of elimination:

1. Elimination by Addition.
2. Elimination by Comparison.
3. Elimination by Substitution.

ELIMINATION BY ADDITION

The simultaneous equations

$$2x+3y=23 \text{ and } 2x-3y=5,$$

may be combined by addition into the equation, $4x=28$, or into the equation $6y=18$, by subtracting the second from the first. Each of the resulting equations contains but one unknown number.

In the simultaneous equations

$$2x+3y=23 \text{ and } 3x-2y=15,$$

the coefficients either of x or of y must be made the same, in order to eliminate one of the two unknown numbers.

METHOD

Given $2x+3y=23$ (*a*) $3x-2y=15$ (*b*).

I. To eliminate y,

Multiply (*a*) by 2, $4x+6y=46$ (*c*)

Multiply (*b*) by 3, $\underline{9x-6y=45}$ (*d*)

Add (*c*) and (*d*), $13x=91$

This gives $x=7$

Substituting this value of x in (*a*) it becomes $14+3y=23$.

Transposing, $3y=9$, $\therefore$ $y=3$.

II. To eliminate x,

Multiply (*a*) by 3, $6x+9y=69$ (*e*)

Multiply (*b*) by -2, $\underline{-6x+4y-30}$ (*f*)

Adding (*e*) and (*f*) $13y=39$

This gives $y=3$.

In order that the coefficient of x in (*f*) may have a negative coefficient, (*b*) is multiplied by -2.

WRITTEN EXERCISES

1. $2x+3y=18$. $3x-2y=1$.
2. $3x-4y=7$. $4x+5y=30$.
3. $4x+y=38$. $x-4y=1$.
4. $5x-7y=0$. $3x+4y=41$.

ELIMINATION BY COMPARISON

Given $3x + 2y = 13$, (a), and $4x - 3y = 6$, (b), to find the value of x and of y.

METHOD

Transposing $2y$ in (a),

$$3x = 13 - 2y$$

$$\therefore x = \frac{13 - 2y}{3} \quad (c)$$

Transposing $-3y$ in (b),

$$4x = 6 + 3y$$

$$\therefore x = \frac{6 + 3y}{4} \quad (d)$$

The second members of (c) and (d) being each equal to x, they are equal to each other; that is,

$$\frac{13 - 2y}{3} = \frac{6 + 3y}{4} \quad (e)$$

Clearing of fractions,

$$52 - 8y = 18 + 9y.$$

Transposing, $-8y - 9y = 18 - 52.$

Combining, $-17y = -34.$

Dividing by -17, $y = 2$. *Ans.*

Substituting this value of y in (a) the latter becomes,

$$3x + 4 = 13.$$

Transposing $3x = 13 - 4 = 9,$

$$\therefore x = 3. \quad \textit{Ans.}$$

Test : When $x \equiv 3$ and $y \equiv 2$, the first member of (b) becomes $4 \cdot 3 - 3 \cdot 2 \equiv 12 - 6 \equiv 6.$

Note that the results are tested by substituting them in (b), since (a) has been used to determine the value of x by substituting in it the value found for y.

This method of elimination is called *elimination by comparison.* The value of one of the two unknown numbers is determined from each equation in terms of the other number, and these two values are made an equation containing but one unknown number.

WRITTEN EXERCISES

Find the roots:

1. $5x + y = 26, \quad 4x - 8y = 12.$
2. $8x + 10y = 4, \quad 12x - 5y = 5.$
3. $4x + 7y = 23, \quad 6x + 5y = 18.$
4. $4x + 7y = 47, \quad 11x = 6y + 3.$
5. $x + y = 17, \quad 4x + 3y = 63.$
6. $3x + 5y = 22, \quad 4x + 7y = 30.$
7. $2x + 5y = 69, \quad 10x - 12y = -62.$
8. $7x + 11y = 125, \quad 8x - 23y = -35.$
9. $7x - 3y = -16, \quad 2x + 5y = 54.$
10. $6x + 5y = 112, \quad 7x - 2y = 68.$
11. $\frac{x}{2} - \frac{y}{3} = 0, \quad \frac{x}{4} + \frac{y}{9} = 5.$ First clear of fractions.
12. $\frac{3x}{5} - y = 11, \quad x + \frac{y}{5} = 37.$
13. $\frac{3x}{7} - \frac{y}{3} = 5, \quad x + 8y = 196.$
14. $11x + \frac{5y}{6} = 32, \quad x + 11y = 134.$
15. $\frac{x}{2} + \frac{y}{3} = 8, \quad 5x - \frac{3y}{4} = 31.$
16. $\frac{7x}{6} + \frac{5y}{3} = 34, \quad \frac{7x}{8} + \frac{y}{8} = 12.$
17. $\frac{x+y}{2} - \frac{x-y}{6} = 16, \quad \frac{x}{3} + \frac{x+y}{3} = 14.$
18. $\frac{3x-5y}{2} + 3 = \frac{2x+y}{5}, \quad 8 - \frac{x-2y}{4} = \frac{x}{2} + \frac{y}{3}.$
19. $\frac{x}{7} - \frac{9y}{10} = -28, \quad \frac{x}{4} + 2y = 94.$
20. $\frac{3x}{8} - \frac{2y}{9} = 5, \quad 2x + y = 66.$

SUPPLEMENTARY EXERCISES

Find value of unknown numbers.

21. $\frac{1}{x}+\frac{2}{y}=\frac{13}{15}, \frac{2}{x}+\frac{3}{y}=\frac{7}{5}.$

SOLUTION

Given $\frac{1}{x}+\frac{2}{y}=\frac{13}{15}$ (*a*) and $\frac{2}{x}+\frac{3}{y}=\frac{7}{5}$ (*b*).

Multiplying (*a*) by 2, $\frac{2}{x}+\frac{4}{y}=\frac{26}{15}$ (*c*).

Multiplying (*b*) by -1, $-\frac{2}{x}-\frac{3}{y}=\left(-\frac{7}{5}\equiv\right)\frac{21}{15}$ (*d*).

Adding (*c*) and (*d*), $\frac{1}{y}=\left(\frac{5}{15}\equiv\right)\frac{1}{3}.$

Clearing of fractions, $y=3$ *Ans.*

Substituting this value of y in (*a*),

$$\frac{1}{x}+\frac{2}{3}=\frac{13}{15}.$$

Transposing, $\frac{1}{x}=\left(\frac{13}{15}-\frac{2}{3}\equiv\frac{3}{15}\equiv\right)\frac{1}{5}.$

Clearing of fractions, $x=5$ *Ans.*

Test: When $x\equiv 5$ and $y\equiv 3$ the first member of (*b*) becomes $\frac{2}{5}+\frac{3}{3}\equiv\frac{2}{5}+1\equiv\frac{7}{5}.$

In the following examples eliminate by addition, postponing the "clearing of fractions" until the end.

22. $\frac{1}{x}+\frac{2}{y}=1, \frac{2}{x}+\frac{3}{y}=1\frac{3}{4}.$

23. $\frac{3}{x}+\frac{5}{y}=19, \frac{7}{x}-\frac{2}{y}=17.$

24. $\frac{7}{x}+\frac{5}{y}=29, \frac{8}{x}-\frac{3}{y}=7.$

25 $\frac{1}{x}+\frac{3}{y+1}=\frac{5}{4}, \frac{4}{x}-\frac{7}{y+1}=\frac{1}{4}.$

26. $\frac{1}{x+y}+\frac{2}{y}=7\frac{1}{5}, \frac{2}{x+y}-\frac{3}{y}=-6\frac{3}{5}.$

MISCELLANEOUS EXAMPLES

27. $14x - 4y = 1,$
$\frac{6x-1}{5} = 2x - \frac{y}{4}.$

28. $\frac{7x+10y}{3} = 164,$
$\frac{7x}{4} + \frac{3y}{2} = \frac{5y}{4} + 69.$

29. $\frac{1}{x+y} + \frac{1}{x-y} = \frac{96}{7},$
$\frac{1}{x+y} - \frac{1}{x-y} = -\frac{72}{7}.$

30. $3x + 2y = 13xy,$
$2x + 3y = 12xy.$

31. $x + 1 : y :: 5 : 3,$
$\frac{2y}{3} - \frac{5-x}{2} = 6.$

32. $4(4y - 1) = 3x + 2y,$
$\frac{2}{5}(x + 4) - y = 8\frac{1}{2}.$

33. $\frac{x}{2} - 12 = \frac{y}{4} + 8,$
$\frac{x+5}{5} - \frac{2y-x}{4} = 28 - \frac{x}{3}.$

34. $\frac{x+y}{3} - \frac{x-y}{5} = \frac{14}{15},$
$\frac{x-y}{3} + 1 = \frac{x+y}{5} + \frac{13}{15}.$

35. $9x + 8y = 43xy,$
$8x + 9y = 42xy.$

36. $x + 2 : 7 :: y + 3 : 9,$
$\frac{x+2y}{9} - \frac{y-2x}{7} + \frac{1}{21} = 6.$

WRITTEN PROBLEMS

1. Find two numbers, the sum of $\frac{3}{7}$ of the first and $\frac{2}{5}$ of the second being 31, and the remainder after taking $\frac{4}{5}$ of the second from $\frac{2}{7}$ of the first being 10.

2. A had 320 acres of land and B had 200. After A had sold x acres and B had sold y acres, the former had twice as much land remaining as the latter. Had A bought x acres and B y acres, instead of selling these quantities, A would have had $1\frac{1}{2}$ times as much land as B. How many acres were sold by each?

3. If B should sell 160 acres of his land to A, the latter would then have 6 times as much land as B. If A should sell B 240 acres, B would then have 6 times as much as A. How many acres has each?

$$x + 160 = 6(y - 160);$$
$$y + 240 = 6(x - 240).$$

4. A man bought two horses and a double set of harness. The first horse together with the harness cost twice as much as the second horse. The second horse with the harness cost $1\frac{1}{3}$ times as much as the first horse. If the harness cost \$125, find the cost of each horse.

5. One third of the difference between two numbers is equal to 10 less than their sum, and the difference between their sum and their difference is two thirds of the larger number. What are the numbers?

6. A farm is divided into two portions, $\frac{3}{7}$ of the first being equal to $\frac{2}{5}$ of the second. The difference between $\frac{3}{4}$ of the first and $\frac{2}{3}$ of the second is 5 acres. How many acres in the farm?

7. The difference of two numbers is equal to their product, and their sum is equal to five times their product. Find the numbers.

8. A man bought 90 yards of calico of two grades, 2 dollars' worth of each grade; at another time he bought at the same rate 3 dollars' worth of one grade and 5 dollars' worth of the other, receiving 175 yards. What was the price per yard of each grade?

$$\frac{200}{x}+\frac{200}{y}=90, \textit{ etc.}$$

9. One day a boy sold 76 bananas and oranges for \$1.20, the receipts for each variety being the same. The next day his sales were again \$1.20 although the number sold was 1 less, but the receipts for the two varieties bore the ratio of 3 to 5, respectively. Find the selling price of each variety.

$$\frac{60}{x}+\frac{60}{y}=76, \textit{ etc.}$$

10. When A works 3 days and B works 5 days, they do $\frac{7}{12}$ of a piece of work. When A works 5 days and B works 3 days, they do $\frac{37}{60}$ of the work. How long would each require to do the whole work?

11. If Peter gives Paul 7 apples, each will have the same number. If Paul gives 5 to Peter, the latter will have three times as many as the former. How many apples has each?

12. When 5 is added to each term of a fraction, its value becomes $\frac{4}{5}$; when 5 is subtracted from each term, its value becomes $\frac{1}{2}$. Find the fraction.

13. A debt of $54 is paid with 15 bills of the denominations of $2 and $5. How many of each denomination are employed?

14. In paying the sum of $414, 20 bills of one denomination and 7 of another are employed. To pay the sum of $136, 6 bills of the former denomination are used and 8 of the latter. What is the denomination of each?

15. A farm containing 1262 acres is divided into two portions, the difference between them being 29 acres more than one fourth the number of acres in the smaller. How many acres are there in each portion?

16. A number is $20\frac{1}{2}$ times the sum of its digits. When it is increased by 396, the order of its digits is inverted. The difference between the first and second digits is equal to that between the second and third. Find the number.

Let $y =$ difference,
and $x =$ hundreds' digit,
then $x + y =$ tens' digit,
and $x + 2y =$ ones' digit.

17. If 6 acres were taken from M's field and added to N's, N's field would then be $2\frac{1}{2}$ times as large as M's. If 6 acres were taken from N's field and added to M's, M's field would then be $1\frac{1}{8}$ times as large as N's. How many acres are there in each field?

18. By making a picture 6 inches longer and 4 inches wider, its area is increased 364 square inches. By making it 4 inches shorter and 6 inches narrower, its area is decreased 336 square inches. What are its dimensions?

SUPPLEMENTARY WORK

THREE UNKNOWN NUMBERS

To find the respective values of three unknown numbers, three simultaneous equations are required.

Find the roots of the simultaneous equations:

$$x + y + 2z = 41,\ x + 3y + z = 47, \text{ and } 6x + 4y + 3z = 120.$$

SOLUTION

Given
$$\begin{aligned} x + y + 2z &= 41 \quad (a)\\ x + 3y + z &= 47 \quad (b)\\ 6x + 4y + 3z &= 120 \quad (c) \end{aligned}$$

Multiplying (a) by -1, $\quad -x - y - 2z = -41 \quad (d)$

(b) $\quad x + 3y + z = 47 \quad (b)$

Adding (d) and (b), $\quad 2y - z = 6 \quad (e)$

Multiplying (c) by -1, $\quad -6x - 4y - 3z = -120 \quad (f)$

Multiplying (a) by 6, $\quad 6x + 6y + 12z = 246 \quad (g)$

Adding (f) and (g), $\quad 2y + 9z = 126 \quad (h)$

Multiplying (e) by -1, $\quad -2y + z = -6 \quad (i)$

(h) $\quad 2y + 9z = 126 \quad (h)$

Adding (i) and (h), $\quad 10z = 120$

Therefore $\quad z = 12$ *Ans.*

Substituting 12 for z in (e), $2y - 12 = 6$, $y = 9$ *Ans.*

Substituting 9 and 12, for y and z, respectively, in (a),

$x + 9 + 24 = 41$, $x = 8$ *Ans.*

Test: When $x \equiv 8$, $y \equiv 9$, $z \equiv 12$, the first members of (b) and (c) become, respectively,

(b) $x + 3y + z \equiv 8 + 3 \cdot 9 + 12 \equiv 8 + 27 + 12 \equiv 47.$

(c) $6x + 4y + 3z \equiv 6 \cdot 8 + 4 \cdot 9 + 3 \cdot 12 \equiv 48 + 36 + 36 \equiv 120.$

To find the roots of three simultaneous equations, eliminate one unknown number by combining two of the given equations, and eliminate the same unknown number by combining the remaining given equation with one of the other two. Combine the resulting equations into an equation containing one unknown number.

WRITTEN EXERCISES

Find the values of the unknown numbers.

1. $$\begin{aligned}4x+3y-2z&=21,\\ 9x-5y+3z&=27,\\ 2x+y-z&=9.\end{aligned}$$

2. $$\begin{aligned}x+2y-z&=15,\\ 4x-y+3z&=34,\\ 3x-2y+2z&=17.\end{aligned}$$

3. $$\begin{aligned}8x-4y+z&=24,\\ 6x+y-z&=84,\\ x-3y-4z&=-80.\end{aligned}$$

4. $$\begin{aligned}6x+4y-3z&=23,\\ 4x-3y+6z&=12,\\ 3x+6y-4z&=17.\end{aligned}$$

5. $$\begin{aligned}7x+4y-2z&=72,\\ 5x-5y-3z&=-13,\\ 2x-3y-4z&=-29.\end{aligned}$$

6. $$\begin{aligned}3x-y-2z&=0,\\ 6x+2y+3z&=45,\\ 4x+3y-z&=31.\end{aligned}$$

7. $x+99=y+z,\quad y+99=2x+2z,\quad z+99=3x+3y.$

Clear of fractions. Eliminate:

8. $$\begin{aligned}x+\frac{y}{2}+\frac{z}{2}&=50,\\ y+\frac{x}{2}+\frac{z}{2}&=52,\\ z+\frac{x}{2}+\frac{y}{2}&=58.\end{aligned}$$

9. $$\begin{aligned}x+\frac{y+z}{2}&=55,\\ y+\frac{x+z}{3}&=50,\\ z+\frac{x+y}{5}&=50.\end{aligned}$$

10. $\frac{x}{2}+\frac{y}{3}+z=13,\quad 2x-3y+\frac{z}{2}=4,\quad \frac{x+2y}{2y+z}=\frac{5}{8}.$

In the examples 11, 12 and 13, add together the three equations, and from one half of their sum deduct each equation separately. In examples 13 and 14 postpone the clearing of fractions until the end.

11. $$\begin{aligned}x+y&=7,\\ x+z&=8,\\ y+z&=9.\end{aligned}$$

12. $$\begin{aligned}x+y&=\tfrac{5}{6},\\ y+z&=\tfrac{7}{12},\\ x+z&=\tfrac{3}{4}.\end{aligned}$$

13. $$\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{5}{6},\\ \frac{1}{y}+\frac{1}{z}&=\frac{7}{12},\\ \frac{1}{x}+\frac{1}{z}&=\frac{3}{4}.\end{aligned}$$

14. $$\begin{aligned}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=9,\\ \frac{2}{x}+\frac{3}{y}+\frac{4}{z}&=29,\\ \frac{3}{x}+\frac{5}{y}-\frac{2}{z}&=13.\end{aligned}$$

WRITTEN PROBLEMS

1. The sum of the first of three numbers and the half of the other two is 25, the sum of the second and the half of the other two is 26, the sum of the third and the half of the other two is 29. Find the numbers.

2. A's money added to one half of B's and C's would amount to $ 155, B's money added to one third of A's and C's would amount to $ 140, and C's money added to one fourth of A's and B's would amount to $ 120. How much money had each?

3. A farmer has three fields. The first contains 15 acres less than the combined areas of the other two, the second contains 25 acres less than the combined areas of the other two, and the third contains 35 acres less than the combined areas of the other two. How many acres are there in each field?

4. The sum of the first of three numbers and one half the second is 14, the sum of the second and one third of the third is 17, the sum of the third and one fourth of the first is 17. Find the numbers.

5. A drover paid $ 75 for 10 sheep, 3 hogs, and a cow; at the same rates he could have bought 3 sheep, 7 hogs, and 2 cows for $ 104; or 7 sheep, 5 hogs, and 3 cows for $ 136. How much did he pay for each?

CHAPTER V

EVOLUTION AND INVOLUTION

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Evolution of Monomials

By the evolution of a term is meant the employment of the term as a factor a number of times. The product is called a **power**. The number indicating how many times the term is to be employed as a factor is called the **index** of the power.

Thus,
$$(a)^5 = a \cdot a \cdot a \cdot a \cdot a = a^5,$$
$$(a^2b)^4 = a^2b \cdot a^2b \cdot a^2b \cdot a^2b = a^8b^4,$$
$$(3\,m^2y^3)^2 = 3\,m^2y^3 \cdot 3\,m^2y^3 = 9\,m^4y^6.$$

In the expressions, $(a)^5$, $(a^2b)^5$, etc., 5 is the index, and shows that the term within the parenthesis is to be raised to the 5th power; in the expression a^5, $a^{10}b^5$, the 5 is called an **exponent**, and indicates that the letter to which it is affixed has been raised to the 5th power. The distinction is not very material.

To raise a monomial to any power, raise the numerical coefficient to the required power and multiply each of the exponents of the letters by the index of the required power.

NOTE. Every odd power of a monomial will have the same sign as that of the quantity from which it is formed. Every even power will be positive.

SIGHT EXERCISES

Square the following:

1. $3ab$.
2. $4mn^2$.
3. $5xy$.
4. $\frac{y}{3z}$.
5. $\frac{4a^3b}{7m^3n^2}$.
6. $-2a^2d^3$.
7. $-9bd^2$.
8. $-12m^2n^2$.
9. $-\frac{2ab}{3cd}$.
10. $-\frac{3a^2m}{4bc}$.
11. $\pm 7x^4y$.
12. $\pm 11a^3x^2$.
13. $\pm 8z^3$.
14. $\pm\frac{4x^2y}{5yz^2}$.
15. $\pm\frac{6a^3b}{7x^2y^3}$.

Give answers:

16. $(2a^2b)^3$.
17. $(2xy)^4$.
18. $(2mn)^5$.
19. $\left(\frac{x^2}{3}\right)^3$.
20. $\left(\frac{2x^2}{3}\right)^4$.
21. $(5xy^2)^2$.
22. $(4mn^2)^3$.
23. $(2a^2b)^4$.
24. $\left(\frac{y^2}{2z}\right)^5$.
25. $\left(\frac{4a^2b}{7m^3n^2}\right)^2$.
26. $(-2ab^2)^3$.
27. $(-2xy)^4$.
28. $(-2mn^2)^5$.
29. $\left(-\frac{x^2}{3}\right)^3$.
30. $\left(-\frac{2a^2}{3}\right)^4$.
31. $(-2a^2c)^5$.
32. $(-5bd^2)^3$.
33. $(-12m^2n^2)^2$.
34. $\left(-\frac{2ab}{3cd}\right)^4$.
35. $\left(-\frac{3a^2m}{4bc^2}\right)^3$.
36. $[(2a)^2]^3$.
37. $[(2a)^3]^2$.
38. $(2a)^6$.
39. $\left[\left(\frac{2a}{3}\right)^2\right]^3$.
40. $\left[\left(\frac{2a}{3}\right)^3\right]^2$.
41. $(9x^3y)^3$.
42. $(\pm 11a^3x^2)^2$.
43. $(\pm 10x^2y)^4$.
44. $\left(-\frac{4x^2y}{9z^2}\right)^3$.
45. $\left(\pm\frac{a^2b}{2xy^2}\right)^6$.

Involution of Monomials

To obtain a root of a monomial extract the required root of the numerical coefficient and divide each of the exponents of the letters by the index of the required root.

The odd root of a monomial will have the same sign as that of the quantity whose root is extracted. The even root of a positive quantity may be either positive or negative.

SIGHT EXERCISES

Give the positive square roots of the following:

1. $9\,a^2b^2$.
2. $25\,x^2y^4$.
3. $16\,m^2n^6$.
4. $49\,c^4d^6$.
5. $\dfrac{16\,a^4b^2}{49\,m^6n^4}$.
6. $36\,a^2y^4$.
7. $100\,b^4z^6$.
8. $64\,c^2m^6$.
9. $121\,n^4p^8$.
10. $\dfrac{81\,a^2b^6}{400\,c^2}$.
11. $144\,x^{10}$.
12. $81\,x^8y^2$.
13. $49\,m^4z^6$.
14. $36\,n^4y^6$.
15. $\dfrac{64\,a^2x^2}{125\,b^4y^4}$.
16. $25\,c^4d^4$.
17. $9\,m^6n^4$.
18. $16\,y^4z^{10}$.
19. $64\,x^2y^8$.
20. $5\frac{4}{9}$.

Give the root of each as indicated:

21. $\sqrt[3]{-64\,m^6n^9}$.
22. $\sqrt{144\,a^4b^6}$.
23. $\sqrt[5]{-243\,a^{10}}$.
24. $\sqrt{\sqrt{16\,a^4b^8}}$.
25. $\sqrt[7]{\dfrac{a^7b^{14}}{128}}$.
26. $\sqrt[6]{64\,a^6b^{12}}$.
27. $\sqrt[5]{-32\,x^{10}}$.
28. $\sqrt{100\,m^2n^4}$.
29. $\sqrt[3]{\sqrt{64\,x^6}}$.
30. $\sqrt[4]{81\,m^4n^8}$.
31. $\sqrt[3]{-27\,a^6x^3}$.
32. $\sqrt[4]{\dfrac{16\,a^4b^8}{81\,c^8d^{12}}}$.
33. $\sqrt[3]{-729\,b^9}$.
34. $\sqrt{\sqrt[3]{64\,n^6}}$.
35. $\sqrt{(75)^2}$.
36. $\sqrt[3]{(56)^3}$.
37. $\sqrt[4]{(81)^2}$.
38. $\sqrt[3]{8\cdot 8}$.
39. $\sqrt[5]{-\dfrac{y^{10}}{32\,z^5}}$.
40. $\sqrt[6]{(27)^2}$.
41. $\sqrt[6]{(\frac{9}{25})^3}$.

Squares of Binomials

SIGHT EXERCISES

Give answers:

1. $(2\,a+3\,b)^2=(2\,a)^2+2(2\,a)(3\,b)+(3\,b)^2$.
2. $(3\,m^2-7\,n)^2=(3\,m^2)^2-2(3m^2)(7\,n)+(7\,n)^2$.
3. $(a+2\,b)^2$.
4. $(2\,x+y)^2$.
5. $(x+7)^2$.
6. $(ab+c)^2$.
7. $(4\,m+n)^2$.
8. $(9+y)^2$.
9. $(6+x^2)^2$.
10. $(x^2-y^2)^2$.
11. $(x^2+x)^2$.
12. $(m+3\,n)^2$.
13. $(a^3+4)^2$.
14. $(a^2+2\,n)^2$.
15. $(c+5)^2$.
16. $(c^2+5\,d)^2$.
17. $(8+m)^2$.
18. $(8-n)^2$.
19. $(x^3+y^3)^2$.
20. $(x^3-x^2)^2$.

Squares of Arithmetical Numbers

Find the square of 73.

$$73^2 \equiv (70+3)^2 \equiv 70^2 + 2 \cdot 70 \cdot 3 + 3^2$$
$$\equiv 4900 + 420 + 9$$
$$\equiv 5329. \quad \textit{Ans.}$$

SIGHT EXERCISES

Square the following:

1. 15.
2. 25.
3. 35.
4. 45.
5. 55.
6. 65.
7. 75.
8. 85.
9. 95.
10. 51.
11. 61.
12. 71.
13. 81.
14. 91.
15. 99.
16. 89.
17. 79.
18. 69.
19. 59.
20. 49.
21. 39.
22. 29.
23. 19.
24. 22.
25. 32.
26. 42.
27. 52.
28. 62.
29. 72.
30. 82.

Square Roots of Trinomials

Since the square of a binomial consists of the sum of the squares of the terms added to twice their algebraic product, an inspection of a trinomial that is a complete square, shows the square root. The two terms of the latter are the square roots of two terms of the trinomial, their connecting sign being the sign of the remaining term of the trinomial.

SIGHT EXERCISES

Give the square roots:

1. $a^2 + b^2 - 2ab$.
2. $m^2 + 4n^2 + 4mn$.
3. $x^2 - 8x + 16$.
4. $a^2b^2 + 2abc + c^2$.
5. $x^2 + 49 - 14x$.
6. $16m^2 + 8mn + n^2$.
7. $y^2 + 81 - 18y$.
8. $12x^2 + x^4 + 36$.
9. $a^2 + 9b^2 + 6ab$.
10. $25 + 10c + c^2$.
11. $m^2 + 64 - 16m$.
12. $4a^2 + 9b^2 + 12ab$.
13. $25x^2 + 4a^2 - 20ax$.
14. $16y^2 - 40y + 25$.
15. $9n^2 + 25 - 30n$.
16. $81 + 16x^2 - 72x$.
17. $9a^2 - 24a + 16$.
18. $49 + 9y^2 - 42y$.

Square Root of Arithmetical Numbers

Extract the square root of 5329.

ANALYSIS

$$\begin{array}{rr|l}
 & 5329 & \underline{70+3} \\
 & 4900 \equiv 70^2 & \\
\text{Trial divisor } 140) & 429 & \\
 & \underline{420} \equiv 2 \cdot 70 \cdot 3 & \\
 & 9 & \\
 & \underline{9} \equiv 3^2. &
\end{array}$$

The process of finding the square root of 5329 consists in decomposing it into the square of the tens, twice the product of the tens by the ones, and the square of the ones.

The square of the tens is clearly 4900, which leaves 429 for twice the product of the tens by the ones, plus the square of the ones. The tens being 70, twice 70, or 140, is written as a trial divisor. Since this is contained 3 times in 429, 3 is taken as the ones, and its product by 140 is deducted from 429, leaving a remainder of 9. As this is the square of 3, the ones' figure is 3 and the square root of 5329 is 73.

In practice, the operation is shortened as follows:

PROCESS

First Step

$$\begin{array}{rr|l}
 & 53'29 & \underline{7} \\
 & \underline{49} & \\
14 & 4\,29 &
\end{array}$$

Completed Work

$$\begin{array}{rr|l}
 & 53'29 & \underline{73} \ \textit{Ans.} \\
 & \underline{49} & \\
143 & 4\,29 & \\
 & \underline{4\,29} &
\end{array}$$

Beginning at the ones' figure of 5329, two places are marked off. The largest square in the first period, 53, is written underneath it, and its root, 7, is placed at the left. To the remainder, 4, the next period, 29, is annexed. For a trial divisor, 14, which is twice 7, is written at the left. Since 14 (tens) is contained 3 times in 42 (tens), the first two figures of 429, 3 is placed after 7 in the root and also after 14 to complete the divisor. The product of the latter (143) by 3 is then written. This also being 429, there is no remainder, and 73 is the root required.

By this method, the sum of $2 \cdot 70 \cdot 3 + 3^2$ is obtained at once by multiplying $(2 \cdot 70 + 3)$ by 3.

WRITTEN EXERCISES

Find the square roots of the following:

1. 289.
2. 625.
3. 576.
4. 529.
5. 2809.
6. 6724.
7. 6084.
8. 3136.
9. 3844.
10. 8649.
11. 2025.
12. 1156.
13. 5041.
14. 7921.
15. 4489.
16. 1936.

Extract the square root of 823.69.

PROCESS

```
        8'23'.69|28.7  Ans.
        4
  48    4 23
        3 84
 567     39 69
         39 69
```

Commencing at the decimal point, separate the whole number into periods of two places each. The first divisor, 48, is obtained by doubling 2, the first figure of the root and annexing the second figure, 8, to the product. The second divisor is obtained by doubling 28, the first two figures of the root, and annexing to 56, the product, 7, the third figure of the root.

17. 20.25.
18. 29.16.
19. 65.61.
20. 13.69.
21. .4356.
22. .5041.
23. .015129.
24. .054756.

When a decimal contains an odd number of places, a cipher must be annexed to make the number of places *even*.

Thus the square root of .9 is obtained by changing it to .90; that is, making it 90 hundredths.

Give the square root of the following to three decimal places:

25. .9
26. .5
27. .75
28. 1.6

SIGHT EXERCISES

Extract the square root of the following:

1. $\frac{4}{9}$.
2. $2\frac{1}{4}$.
3. $\frac{16}{25}$.
4. $1\frac{9}{16}$.
5. $\frac{25}{36}$.
6. $1\frac{11}{25}$.
7. $\frac{9}{49}$.
8. $5\frac{4}{9}$.
9. $\frac{49}{100}$.
10. $2\frac{2}{49}$.

Square Roots of Polynomials

WRITTEN EXERCISES

Extract the square roots:

1. $29x^2 - 30x + 25 + 4x^4 - 12x^3$.

Process

Given $29x^2 - 30x + 25 + 4x^4 - 12x^3$.

Arrange the expression according to the descending powers of x.

$$\begin{array}{lll} & 4x^4 - 12x^3 + 29x^2 - 30x + 25 & \big|\underline{2x^2 - 3x + 5} \quad \textit{Ans.} \\ & \underline{4x^4} & \\ 4x^2 - 3x & -12x^3 + 29x^2 & \\ & \underline{-12x^3 + 9x^2} & \\ 4x^2 - 6x + 5 & 20x^2 - 30x + 25 & \\ & \underline{20x^2 - 30x + 25} & \end{array}$$

The square root ($2x^2$) of the first term ($4x^4$) is written as the first term of the root and its square ($4x^4$) is deducted from the given polynomial. As a matter of convenience, two terms ($-12x^3 + 29x^2$) are brought down, and twice $2x^2$, the first term of the root, is written to the left as a trial divisor. The quotient ($-3x$) of the first term of the partial remainder, $-12x^3$, by the trial divisor is algebraically added to the trial divisor and to the first term of the root. The new divisor ($4x^2 - 3x$) is multiplied by $-3x$, the product written under the partial remainder and the difference is brought down, together with the remaining terms of the given expression. The first two terms of the root are doubled and written as a second partial divisor, which is completed by 5, the quotient of $20x^2$, the first term of the final remainder, by $4x^2$, the first term of the second divisor. The product of the second divisor by 5 being equal to the remainder, the root is $2x^2 - 3x + 5$.

2. $x^4 - 4x^3 + 10x^2 - 12x + 9$.

3. $9x^6 - 30x^5 + 67x^4 - 70x^3 + 49x^2$.

4. $a^4x^4 + 2a^2bx^3 - 2a^2cx^2 + b^2x^2 - 2bcx + c^2$.

5. $16m^2n^2 - 16m^2n + 4m^2 + 24mn^2 - 12mn + 9n^2$.

RADICALS

A radical is an indicated root, the small figure, called the index, and placed at the top of the left arm of the sign, showing the root that is to be extracted. The index is omitted in the case of square root.

Thus $\sqrt{9}$ indicates the square root of 9, sometimes read "radical" 9; $\sqrt[3]{27}$ indicates the cube root of 27; $\sqrt[4]{625}$ indicates the fourth root of 625.

A radical is simplified by extracting the indicated root, when possible.

$$\sqrt{9} \equiv \pm 3; \quad \sqrt[3]{27} \equiv 3; \quad \sqrt[4]{625} \equiv \pm 5.$$

When the quantity under the radical sign contains a factor whose indicated root can be extracted, the radical is simplified by extracting the indicated root of this factor and prefixing it to the radical sign under which is written the remaining factor.

Thus
$$\begin{aligned}\sqrt{8} &\equiv \sqrt{4 \times 2} \equiv 2\sqrt{2}\\ \sqrt[3]{108} &\equiv \sqrt[3]{27 \times 4} \equiv 3\sqrt[3]{4}\\ \sqrt[4]{a^5b^9} &\equiv \sqrt[4]{(a^4b^8)(ab)} \equiv ab^2\sqrt[4]{ab}.\end{aligned}$$

Radicals are said to be **like** or **similar** when they have the same index and the same quantity under the radical sign.

Thus, $3\sqrt{2}$ and $\sqrt{2}$ are similar; so are $2\sqrt[3]{3}$ and $3\sqrt[3]{3}$; *etc.*

Like radicals may be added by writing the algebraic sum of their coefficients before the common radical.

$$3\sqrt{2} + \sqrt{2} \equiv 4\sqrt{2}; \quad 2\sqrt[3]{3} - 3\sqrt[3]{3} \equiv -\sqrt[3]{3}.$$

Radicals having the same index are multiplied by writing under the common radical sign the product of the quantities.

$$\sqrt{2} \cdot \sqrt{3} \equiv \sqrt{6}; \quad 2\sqrt{2} \cdot 5\sqrt{3} \equiv 10\sqrt{6}$$
$$4\sqrt[3]{3} \cdot \sqrt[3]{3} \equiv 4\sqrt[3]{9}; \quad 2\sqrt[3]{5} \cdot 3\sqrt[3]{7} \equiv 6\sqrt[3]{35}$$

The results should be simplified, when possible

$$\sqrt{2} \cdot 3\sqrt{2} \equiv 3\sqrt{4} \equiv 6$$
$$4\sqrt[3]{3} \cdot 2\sqrt[3]{9} \equiv 8\sqrt[3]{27} \equiv 24.$$

To raise a radical to the power shown by the index of the root, the radical sign is removed.

Thus x^3 is the square of $\sqrt{x^3}$

2 is the square of $\sqrt{2}$

3 is the cube of $\sqrt[3]{3}$.

In the following examples, the radical term should constitute one member of the equation, and the remaining terms should be transposed to the other if the root of the radical cannot be extracted. The resulting equation is then transformed into one without a radical, by raising both members to the power shown by the index of the radical.

To solve $2x - \sqrt{64} = 0$, the equation is transformed into $2x = \sqrt{64}$, and the square root of the second member extracted, giving the new equation, $2x = \pm 8$; whence, $x = \pm 4$. *Ans.*

To solve $\sqrt{2x} + 1 = 7$.

Transpose, $\sqrt{2x} = 6$.

Square both members, $2x = 36$.

$\therefore x = 18$. *Ans.*

To solve $\sqrt{2x+1} = 7$.

Square both members, $2x + 1 = 49$.

Transpose, $2x = 48$.

$\therefore x = 24$. *Ans.*

SIGHT EXERCISES

Give positive value of x:

1. $x^2 = \sqrt{16}$.
2. $x - \sqrt{9} = 0$.
3. $x = \sqrt{3x}$.
4. $\sqrt{x^2} - \sqrt{4} = 0$.
5. $\sqrt{x^2 - 4} = 0$.
6. $\sqrt{x+1} = 9$.
7. $\sqrt{x+3} + 2 = 5$.
8. $\sqrt{x^2} + \sqrt{9} = 5$.
9. $\sqrt{x^2+9} = 5$.
10. $\sqrt{x^2+9} = x + 1$.

Simplify the following:

11. $\sqrt{2} + \sqrt{8}$.
12. $\sqrt{8} - \sqrt{2}$.
13. $\sqrt{2} \cdot \sqrt{8}$.
14. $\sqrt{8} \div \sqrt{2}$.
15. $4\sqrt{3} \div 2$.
16. $\sqrt{8} = ?\ \sqrt{2}$.

Review Exercises and Problems

SIGHT EXERCISES

Give answers:

1. $[(x+y)+z]^2$.
2. $(a+b)(a+c)$.
3. $[(x+y)+1]^2$.
4. $(a+2)(b+3)$.
5. $\sqrt{4\times 4\times 4}$.
6. $[(a-b)+c]^2$.
7. $(a-b)(a-c)$.
8. $[(a+b)-1]^2$.
9. $(x+3)(y+4)$.
10. $\sqrt{9\times 9\times 9}$.
11. $[m-(n+p)]^2$.
12. $(a+b)(x+y)$.
13. $[x^2-(y-1)]^2$.
14. $(n+2)(n+5)$.
15. $\sqrt{2\times 3\times 6}$.

Solve:

16. $x^2=\sqrt{81}$.
17. $\sqrt{x}+1=3$.
18. $x^2+2=11$.
19. $3x^2=36x$.
20. $x:x^2::1:2$.
21. $x^2-1=\sqrt{64}$.
22. $\sqrt{2x}+1=3$.
23. $2x^2+1=9$.
24. $5x^2=10x$.
25. $x:x^2::3:9$.
26. $x^2-3=\sqrt{36}$.
27. $\sqrt{2x+1}=3$.
28. $2x^2-1=31$.
29. $9x^2=18x$.
30. $x:x^2::1:7$.

SIGHT PROBLEMS

1. Find two numbers whose sum is 38 and whose difference is 10.

2. What three consecutive numbers have a sum of 63?

3. Give two consecutive odd numbers whose sum is 60.

4. Give three consecutive even numbers whose sum is 42.

5. What is the middle number of five consecutive odd numbers whose sum is 55?

6. When 45 is added to a number, its digits are inverted. Find the difference between the digits.

7. In what number will the digits be inverted when it is increased by 72?

8. What two numbers will have their digits inverted when either is increased by 63?

9. In how many numbers will the digits be inverted by the addition of 54 to the number?

WRITTEN EXERCISES

Find the value of x.

1. $(x+1)(x+2)=x^2+14$.
2. $(x+3)(x+5)=x^2+47$.
3. $(x+2)(x+3)=5x+31$.
4. $(x+2)(x+4)=x^2+20$.
5. $(x+3)(x+7)=10x+25$.
6. $(x+1)(x+2)(x-3)=2-7x$.
7. $(x+2)(x+3)(x-5)=x^3-20x$.
8. $(x+3)(x+5)(x-8)=x^3-50x$.
9. $(x+3)(x+7)(x-10)=6-79x$.
10. $(x-2)(x+3)(x+6)=7x^2-9$.
11. $\frac{6}{x}\left(1+\frac{x}{4}+m\right)=\frac{1}{2x}(m+6)12$.
12. $\frac{x+3}{4}-\frac{x-8}{5}-\frac{x-5}{2}+2=1$.
13. $\frac{x}{3}+5+\frac{x}{4}+8+\frac{x}{5}-10=97$.
14. $x-3+\frac{x-3}{2}+\frac{3(x-3)}{4}=2\frac{1}{4}$.
15. $\frac{x+2}{6}-\frac{x-3}{8}+1=\frac{x}{2}-\frac{x-1}{4}$.

WRITTEN PROBLEMS

1 What number subtracted from each of the following, 45, 51, 61, and 71, will form a proportion from the respective remainders?

2. When the digits of a number are three consecutive figures, what number must be added to it in order that the order of the digits may be inverted?

3. A package of money consisting of $27,200 was made up of bills of the following denominations: $1000, $500, $100, $50, $20, $10, $5, $2, $1, the number of bills of each denomination being, respectively, twice, three times, four times, etc., the number of bills of the largest denomination. How many were there of each?

4. How many bills of each of the nine denominations from $1 to $1000 will be required to make $21,360, the number of each denomination being twice of the number of the next higher one?

5. A garrison had rations for eight weeks. Being reenforced, 6000 additional daily rations were used, and the supply lasted only six weeks. If it required 3 rations daily for each soldier, of how many soldiers did the garrison originally consist?

6. How many men must be sent from a garrison of 1500 in order that a six weeks' supply of provisions may last two weeks longer?

7. A contractor agrees to finish a piece of railroad in 105 days. He keeps 210 men at work for 90 days, at the end of which time he finds that one sixth of the task remains to be done. How many additional men are needed to complete the contract on time?

8. A train going 15 miles per hour starts at 6 A.M. At 9 A.M. a train leaves the same place and goes at the rate of 45 miles per hour. How many miles must the second train travel to overtake the first?

9. A man walking 5 miles an hour starts from M one hour ahead of a train going 60 miles an hour. In how many minutes and seconds will the train overtake him?

10. At what time after 1 o'clock will the minute hand overtake the hour hand?

CHAPTER VI

QUADRATIC EQUATIONS

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Quadratic Equations

An equation is said to be **quadratic** when it contains the second power of the unknown number.

An equation that may be simplified into a linear one is not a quadratic equation; such as, $4x^2 = 16x$, which is reducible to $4x = 16$; $x^2 + 2x - 8 = x^2$, which is reducible to $2x - 8 = 0$; $\frac{x^2 - 5x + 6}{x - 3} = 10$, which is reducible to $x - 2 = 10$.

Incomplete Quadratic Equations

A quadratic equation is said to be **incomplete** when it does not contain the first power of the unknown number.

An incomplete quadratic expression is also called a **pure** quadratic.

Every quadratic equation has two roots. When the equation is an incomplete one, the roots are numerically the same but have opposite signs.

One Unknown Number

SIGHT EXERCISES

Give the roots:

1. $x^2 = 9$ *Ans.* $x = \pm 3$.

2. $x^2 = 25$ *Ans.* $x = \pm 5$.

3. $x^2 - 6 = 30$.

4. $2x^2 - 4 = 28$.

5. $3x^2 + 5 = 32$.

6. $\dfrac{x^2+9}{2} = 17$.

7. $4x^2 = 27 + x^2$.

8. $7x^2 - 112 = 0$.

9. $x^2 - 50 = 50$.

10. $\dfrac{2x^2+1}{3} = 3$.

11. $x^2 = 6x - x^2$.

12. $2x^2 - 6x = 0$.

13. $4x^2 - 100 = 0$.

14. $\dfrac{5x^2-12}{4} = 6 + x^2$.

WRITTEN EXERCISES

Find the value of x.

1. $\dfrac{x^2+7}{2} - \dfrac{2x^2-3}{3} = 3$.

Solution

Given $$\frac{x^2+7}{2} - \frac{2x^2-3}{3} = 3.$$

Clearing of fractions,
$$3x^2 + 21 - 4x^2 + 6 = 18.$$

Transposing, $$3x^2 - 4x^2 = 18 - 21 - 6.$$

Combining, $$-x^2 = -9.$$

Multiplying by -1, $$x^2 = 9.$$
$$\therefore x = \pm 3. \text{ Ans.}$$

Test: When $x = \pm 3$ the first member of the original equation becomes $$\frac{9+7}{2} - \frac{18-3}{3} \equiv 8 - 5 \equiv 3.$$

2. $x^2 - \dfrac{16}{3} = \dfrac{7x^2}{15} + 8$.

3. $\dfrac{3x^2+4}{7} = \dfrac{3x^2}{6} - 2$.

4. $\dfrac{2x^2-1}{7} = x^2 - 18$.

5. $\dfrac{x^2-11}{2} = \dfrac{2x^2+33}{5}$.

6. $\dfrac{3x^2-16}{2} + \dfrac{x^2+11}{3} = \dfrac{5x^2+16}{4} + 1$.

7. $\frac{x^2-10}{3}+\frac{x^2+11}{2}=\frac{5x^2-29}{4}-\frac{x^2-13}{12}.$

8. $\frac{2x^2-18}{3}-\frac{x^2+48}{6}=\frac{x^2-6}{2}-\frac{5x^2+18}{18}.$

9. $\frac{x^2+2}{18}-\frac{x^2+4}{10}=\frac{x^2-1}{5}-\frac{x^2+8}{6}.$

10. $\frac{x^2-7}{9}-\frac{x^2-11}{7}=\frac{x^2+17}{21}-\frac{x^2-19}{3}.$

11. $\frac{(x-3)(x+3)}{4}-\frac{(x+2)(x-2)}{9}=\frac{(x+4)(x-4)}{3}$
$-\frac{(x-1)(x+1)}{8}.$

12. $\frac{10x^2-5}{6}=\frac{5x^2-2}{3}-\frac{3x^2-2}{2x^2+4}.$

SOLUTION

Given $$\frac{10x^2-5}{6}=\frac{5x^2-2}{3}-\frac{3x^2-2}{2x^2+4}. \qquad (a)$$

Making the fraction having a binomial denominator the first member of the equation, (a) becomes

$$\frac{3x^2-2}{2x^2+4}=\frac{5x^2-2}{3}-\frac{10x^2-5}{6}. \qquad (b)$$

Combining the fractions in the second member of (b) by reducing them to a common denominator, etc., it becomes

$$\frac{3x^2-2}{2x^2+4}=\left[\frac{10x^2-4-(10x^2-5)}{6}=\right]\frac{1}{6}. \qquad (c)$$

Clearing (c) of fractions,

$$18x^2-12=2x^2+4. \qquad (d)$$

Transposing, $16x^2=16.$

$x^2=1.$

$x=\pm 1.$ *Test.*

NOTE. By the foregoing method an equation is obtained having but one fraction in each member. The equation is then cleared of fractions by cross multiplication.

13. $\dfrac{3x^2-11}{4}-\dfrac{x^2+5}{2x^2-94}=\dfrac{6x^2-130}{8}.$

Suggestion. Make the fraction having a binomial denominator the second member, in order that it may be positive, the new equation being

$$\frac{3x^2-11}{4}-\frac{6x^2-130}{8}=\frac{x^2+5}{2x^2-94}.$$

First combine the two fractional terms in the first member into a single term, thus obtaining an equation having a fractional term for each member.

14. $\dfrac{2x^2-28}{2x^2-42}+\dfrac{3x^2-21}{18}-\dfrac{2x^2+1}{12}=0.$

15. $\dfrac{3x^2+1}{19}-\dfrac{6x^2-74}{38}=\dfrac{x^2+15}{x^2-5}.$

16. $\dfrac{4x^2-16}{x^2+1}=\dfrac{3x^2-7}{5}-\dfrac{6x^2-34}{10}.$

17. $\dfrac{x+4}{x-4}+\dfrac{x-4}{x+4}=\dfrac{10}{3}.$

Combining terms of first member,

$$\frac{(x+4)^2+(x-4)^2}{(x-4)(x+4)}=\frac{10}{3},$$

or, $$\frac{x^2+8x+16+x^2-8x+16}{x^2-16}=\frac{10}{3},$$

or, $$\frac{2x^2+32}{x^2-16}=\frac{10}{3}.$$

Clear of fractions by cross multiplication. Solve.

18. $\dfrac{x-5}{x+5}+\dfrac{x+5}{x-5}=\dfrac{122}{11}.$

19. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}=\dfrac{5}{2}.$

20. $3(x+6)^2+(x-6)^2=4[(x+8)(x+1)-2].$

21. $(3x+6)(3x-6)=(4x-3)(4x+3)-90.$

22. $6[(x+5)^2+(x-5)^2]=37(x+5)(x-5).$

23. $4[(x-4)^2+(x+4)^2]=17(x+4)(x-4).$

24. $5[(x+6)^2+(x-6)^2]=26(x-6)(x+6).$

25. $9[(x-8)^2+(x+8)^2]=82(x+8)(x-8).$

26. $3[(x+7)^2+(x-7)^2]=10(x+7)(x-7).$

WRITTEN PROBLEMS

Note. Give only positive roots as answers.

1. Find two numbers having the ratio of 4 to 3, the sum of their squares being 400.

Let $4x$ and $3x$ represent the numbers.

2. The perimeter of one square is $\frac{7}{8}$ of that of another square, and the difference of their areas is 2000 square yards. What is the side of each?

3. Find the dimensions of a rectangular prism whose dimensions are $2x$, $3x$, and $4x$, respectively, and whose entire surface contains 832 square inches. What is its volume?

4. The product of two consecutive odd numbers is 575. What are the numbers?

Let $x+1$ and $x-1$ represent the numbers.

5. When an officer arranges his men in a square, he finds he has 24 men over. If he increases each side by 1 man, he finds he requires 29 additional men to make a complete square. How many men has he?

$$x^2+24=(x+1)^2-29.$$

6. I exchanged 3 equal square plots and a rectangular one 8 rods by 11 rods for a square plot measuring 26 rods on a side. If the area of the latter is equal to the combined areas of the other plots, what is the length of a side of each small square?

7. The area of a square field contains 10 times as many square rods as there are linear rods in the perimeter. How many acres does the field contain?

8. One side of a rectangular field is $\frac{5}{12}$ the length of the other. The length of the diagonal is 65 rods. Find the dimensions and the area.

9. A man sold a cow for $16 more than its cost, the rate per cent of profit being equal to the number of dollars he paid for the cow. What did the cow cost?

Two Unknown Numbers

In the two simultaneous equations $x^2+y^2=113$ (a) and $x^2-y^2=15$ (b) the sum of (a) and (b) gives $2x^2=128$, from which are obtained $x^2=64$ and $x=\pm 8$. The difference between (a) and (b) gives $2y^2=98$, from which are obtained $y^2=49$ and $y=\pm 7$.

SIGHT EXERCISES

Note. In giving answers, give both values for each unknown number: $x=\pm 4$, $y=\pm 2$.

1. $x^2+y^2=20$,
   $x^2-y^2=12$.
2. $x^2+y^2=26$,
   $x^2-y^2=24$.
3. $x^2+y^2=100$,
   $x^2-y^2=28$.
4. $x^2+y^2=74$,
   $x^2-y^2=24$.
5. $x^2+2y^2=24$,
   $x^2+y^2=20$.
6. $2x^2+y^2=19$,
   $3x^2+y^2=28$.
7. $x^2+y^2=13$,
   $x^2-y^2=5$.
8. $x^2+y^2=149$,
   $x^2-y^2=51$.
9. $2x^2+2y^2=68$,
   $2x^2-y^2=41$.
10. $3x^2+3y^2=30$,
    $3x^2-y^2=26$.

WRITTEN EXERCISES

Find the values of x and of y:

1. $x^2-y^2=64$,
   $9x^2-25y^2=0$.
2. $4x^2+2y^2=38$,
   $9x^2+3y^2=84$.
3. $2x^2+2y^2=20$,
   $9x^2-3y^2=78$.
4. $4x^2+7y^2=148$,
   $9x^2-3y^2=33$.
5. $x^2+y^2=29$,
   $4x^2+7y^2=191$.
6. $x^2+y^2=100$,
   $3x^2+4y^2=336$.
7. $x^2+y^2=34$,
   $10x^2-5y^2=205$.
8. $4x^2+8y^2=96$,
   $3x^2+3y^2=60$.
9. $2x^2-3y^2=5$,
   $5x^2+7y^2=143$.
10. $3x^2+5y^2=30$,
    $x^2+y^2=8\frac{1}{2}$.

11. $2x^2 + 3y^2 = 93$, $5x - 3y = 0$.

Method

Given $2x^2 + 3y^2 = 93$ (*a*)
$\underline{5x - 3y = 0}$ (*b*)
Transposing, (*b*) becomes $5x = 3y$ (*c*)
Squaring (*c*), $25x^2 = 9y^2$ (*d*)
Transposing, $25x^2 - 9y^2 = 0$ (*e*)
Multiplying (*a*) by 3, $\underline{6x^2 + 9y^2 = 279}$ (*f*)
Adding (*e*) and (*f*), $31x^2 = 279$
$x^2 = 9$
$x = \pm 3$

Find values of *y*, and test.

Observe that (*b*) is transformed into (*c*) in order to obtain an equation each member of which consists of a single term.

12. $x^2 + y^2 = 52$,
$3x - 2y = 0$.

13. $x^2 - y^2 = 77$,
$9x - 2y = 0$.

14. $3x^2 - y^2 = 44$,
$4x = 3y$.

15. $2x - y = 0$,
$5x^2 - y^2 = 25$.

16. $2x = 4\frac{2}{3}y$,
$x^2 + y^2 = 58$.

17. $x - 4y = 0$,
$2x^2 - 5y^2 = 108$.

18. $4x - 3y = 0$,
$20x^2 - 9y^2 = 25$,

19. $2x = 3y$.
$9x^2 + 16y^2 = 145$.

WRITTEN PROBLEMS

Note. Give only positive answers.

1. The sum of the squares of two numbers is 117. The difference between them is one third of the greater. Find the numbers.

Solution

$x^2 + y^2 = 117$ (*a*) $x - y = \frac{x}{3}$ (*b*)

Clearing (*b*) of fractions, $3x - 3y = x$.
Transposing and combining, $2x = 3y$.
Squaring, $4x^2 = 9y^2$.
Transposing, $4x^2 - 9y^2 = 0$ (*c*)
Multiplying (*a*) by 9 $\underline{9x^2 + 9y^2 = 1053}$ (*d*)
Add (*c*) and (*d*).

2. The difference of the squares of two numbers is 99, and the quotient of the greater by the less is $1\frac{11}{25}$ times the quotient of the less by the greater. What are the numbers?

$$x^2 - y^2 = 99;\ \frac{x}{y} = \frac{36y}{25x}.$$

3. Find two numbers having 132 for the difference of their squares, their sum being equal to $1\frac{4}{7}$ times the greater.

4. A and B brought to market 15 dozen eggs and each sold them for the same sum. If A had brought as many as B, he would have received 98 cents for them, and if B had brought as many as A, he would have received \$1.28. How many eggs did each bring to market?

Let x = number of dozen brought by A.
y = number of dozen brought by B.
Then $\frac{98}{y}$ = price per dozen received by A.
And $\frac{128}{x}$ = price per dozen received by B.

5. If a certain number, consisting of two digits, is multiplied by the units' digit, the product is 56; if the sum of the digits is multiplied by the same digit, the product is 20. What is the number?

$$(10x + y)\,y = 56.$$
$$(x + y)\,y = 20. \quad \text{Simplify.}$$

Find the value of xy by eliminating y^2. Substitute this value in the second equation.

6. A farmer has two square fields, one of which contains 700 square rods more than the other, and 3 times the perimeter of one is equal to 4 times the perimeter of the other. What is the area of each?

7. The combined area of two square fields is 1872 square rods and the difference between their areas is 720 square rods. Find the dimensions of each.

8. The sum of the squares of two numbers is 130. The product of their sum by their difference is 32. What are the numbers?

Complete Quadratic Equations — One Unknown Number

SIGHT EXERCISES

Give the roots of each of the following equations:

1. $x^2-3x+2=0$.
2. $x^2+x-6=0$.
3. $x^2+x-20=0$.
4. $x^2-8x+15=0$.
5. $x^2-5x-6=0$.
6. $x^2+3x=10$.
7. $x^2-10x=56$.
8. $x^2+x=56$.
9. $x^2-8x=9$.
10. $x^2+7x=18$.

As the roots of the equation $(x+2)(x-3)=0$ are $x=-2$, (*a*), and $x=3$, (*b*), an equation having these roots is formed by changing (*a*) and (*b*) to the form $x+2=0$ and $x-3=0$, respectively, and multiplying the new equations member by member, $(x+2)(x-3)=0$, or $x^2-x-6=0$, or $x^2-x=6$.

SIGHT EXERCISES

Form quadratic equations in which x has values, respectively, as follows:

11. $x=1$, $x=2$.
12. $x=2$, $x=-3$.
13. $x=4$, $x=-5$.
14. $x=3$, $x=5$.
15. $x=-1$, $x=6$.
16. $x=-5$, $x=2$.
17. $x=7$, $x=8$.
18. $x=-8$, $x=7$.
19. $x=9$, $x=-1$.
20. $x=2$, $x=-9$.

WRITTEN EXERCISES

1. $2x^2-46x+264=0$.
2. $4x^2-80x+204=0$.
3. $7x^2+70x-273=0$.
4. $3x^2-12x-231=0$.
5. $5x^2-130x+345=0$.
6. $6x^2-156x+414=0$.
7. $8x^2+168x+304=0$.
8. $5x^2-180x-185=0$.
9. $2x^2+78x-164=0$.
10. $3x^2-75x+138=0$.
11. $4x^2-96x=-572$.
12. $7x^2-154x=-595$.
13. $3x^2+54x=-195$.
14. $5x^2-130x=-345$.

COMPLETING THE SQUARE

A trinomial is a complete square when it contains two terms that are squares and a third term that is twice the product of the square roots of the other two.

To make a complete square of the binomial $x^2 + 36$, twice the product of $\sqrt{x^2}$ by $\sqrt{36}$ must be added; viz. $\pm 2(x \cdot 6)$; or $\pm 12x$, the required expression being $x^2 \pm 12x + 36$.

To make $4x^2 + 25$ a complete square, $\pm(2 \cdot 2x \cdot 5)$; or $\pm 20x$ must be added, which gives the trinomial $4x^2 \pm 20x + 25$.

SIGHT EXERCISES

Complete the squares of the following:

1. $x^2 \pm ? + 25$.
2. $4x^2 \pm ? + 25$.
3. $9x^2 \pm ? + 25$.
4. $16x^2 \pm ? + 25$.
5. $25x^2 \pm ? + 16$.
6. $36x^2 \pm ? + 25$.
7. $49x^2 \pm ? + 25$.
8. $64x^2 \pm ? + 25$.
9. $81x^2 \pm ? + 25$.
10. $100x^2 \pm ? + 81$.

To make a complete square of $9x^2 \pm 24x$ the term to be added is the square of the quotient obtained by dividing $24x$ by twice $\sqrt{9x^2}$; that is, $(24x \div 6x)^2$, or 4^2; or 16.

11. $x^2 + 10x + ?$
12. $x^2 - 20x + ?$
13. $x^2 + 22x + ?$
14. $4x^2 - 20x + ?$
15. $4x^2 + 12x + ?$
16. $4x^2 - 28x + ?$
17. $4x^2 + 36x + ?$
18. $9x^2 - 30x + ?$
19. $9x^2 + 12x + ?$
20. $9x^2 - 42x + ?$
21. $16x^2 + 40x + ?$
22. $16x^2 - 56x + ?$
23. $16x^2 + 72x + ?$
24. $25x^2 - 40x + ?$
25. $25x^2 + 60x + ?$
26. $25x^2 - 80x + ?$
27. $36x^2 + 60x + ?$
28. $36x^2 - 12x + ?$
29. $49x^2 + 70x + ?$
30. $49x^2 - 28x + ?$

It will be observed that the square is "completed" in the last 20 examples by adding the square of the number obtained by dividing one half the coefficient of x by the square root of the coefficient of x^2.

Give the square root of each term.

31. $4x^2+8x+4=9.$
32. $4x^2-12x+9=16.$
33. $4x^2+20x+25=49.$
34. $4x^2-28x+49=81.$
35. $9x^2+12x+4=16.$
36. $9x^2-24x+16=49.$
37. $9x^2+30x+25=64.$
38. $9x^2-42x+49=25.$
39. $16x^2+32x+16=81.$
40. $16x^2-8x+1=49.$
41. $16x^2+8x+1=25.$
42. $16x^2-32x+16=1.$
43. $25x^2+10x+1=9.$
44. $25x^2-20x+4=25.$
45. $25x^2+30x+9=36.$
46. $25x^2-70x+49=64.$
47. $36x^2+132x+121=25.$
48. $36x^2-84x+49=81.$
49. $36x^2+60x+25=49.$
50. $36x^2-12x+1=100.$

WRITTEN EXERCISES

Find the roots of each of the following equations by completing the square.

1. $16x^2+32x=-15.$

Solution

Given $16x^2+32x=-15.$
Completing the square, $16x^2+32x+16=-15+16=1.$
Extracting the square root, $4x+4=\pm 1.$
Transposing, $4x=-3$ or $-5.$
$\therefore x=-\frac{3}{4}$ or $-\frac{5}{4}$. *Ans.*

Test: when $x=-\frac{3}{4}$ the first member becomes
$16(\frac{9}{16})+32(-\frac{3}{4})\equiv 9-24\equiv -15,$
when $x=-\frac{5}{4}$, the first member becomes
$16(\frac{25}{16})+32(-\frac{5}{4})\equiv 25-40\equiv -15.$

2. $4x^2+20x=-9.$
3. $4x^2+8x=5.$
4. $4x^2-12x=7.$
5. $4x^2+20x=24.$
6. $4x^2-28x=32.$
7. $9x^2-12x=5.$
8. $9x^2+12x=12.$
9. $9x^2-24x=33.$
10. $9x^2+30x=39.$
11. $9x^2-42x=-24.$
12. $16x^2+24x=-5.$
13. $16x^2+32x=65.$

14. $16x^2 - 8x = 48.$

15. $16x^2 + 8x = 24.$

16. $16x^2 - 32x = -15.$

17. $25x^2 - 10x = 48.$

18. $25x^2 + 10x = 8.$

19. $25x^2 - 20x = 21.$

20. $25x^2 + 30x = 27.$

21. $9x^2 - 24x = 105.$

22. $9x^2 - 12x = 192.$

23. $25x^2 + 30x = 40.$

24. $25x^2 + 10x = 80.$

25. $25x^2 + 30x = 720.$

26. $49x^2 - 70x = 504.$

27. $49x^2 - 70x = 875.$

28. $49x^2 - 56x = -7.$

29. $36x^2 - 60x = 11.$

30. $36x^2 - 84x = 32.$

31. $36x^2 + 60x = 24.$

32. $36x^2 - 12x = 99.$

33. $49x^2 + 70x = -9.$

34. $64x^2 - 48x = 27.$

35. $36x^2 - 60x = 144.$

36. $16x^2 + 24x = 352.$

37. $100x^2 + 180x = 1600.$

38. $121x^2 + 66x = 2200.$

39. $121x^2 + 110x = 264.$

WRITTEN PROBLEMS

1. The difference between two numbers is 2, and the sum of their squares is 884. What are the numbers?

2. A drover paid $240 for a number of sheep. If he paid $1 apiece less for them, he would have obtained 8 sheep more. How many did he buy?

$$\frac{240}{x-1} - \frac{240}{x} = 8.$$

3. A man sold an article for $56, making as great a profit per cent as the number of dollars in the cost of the article. What did it cost him?

4. The difference between two numbers is 9, and their sum multiplied by the greater is 486. Find the numbers.

5. The owner of a plot of land, the perimeter of which is 3000 feet, finds that he will require for a new fence 50 posts more than were used in the old one, if the posts are placed 3 feet closer together. How far apart were the old posts?

6. A pile of wood 4 feet wide contains 1280 cubic feet. What are the other dimensions if the length is 4 feet more than the height?

7. A farmer has two square fields containing together 976 square rods. Find the area of each if one field is 4 rods longer than the other.

8. The sum of two numbers is 18. If their difference is multiplied by the smaller number, the product is 28. Find the numbers.

9. A walk on the four sides of a square plot 100 feet wide contains 2100 square feet. What is the width of the walk?

$x =$ width,

$\underline{100 + 2x = \text{side}}$ of square including the walk,

$(100+2x)^2 - 100^2 = 2100.$

10. The product of two numbers is 200, and their quotient is 3 with a remainder of 1. Find the numbers.

Let $x =$ smaller,

$3x + 1 =$ greater.

11. The slant height of a square pyramid is 24 inches, and its entire surface is 868 square inches. What is the length of a side of the base?

12. The entire surface of a cone is 462 square inches. Find the radius of the base when the slant height is 14 inches.

13. The surface of a circular frame surrounding a mirror whose exposed diameter is 14 inches, contains $192\frac{1}{2}$ square inches. What is the width of the frame?

14. A and B entered into partnership, B contributing \$3000 and A the remainder. The profits were \$900. What was A's capital if his capital and profits together amounted to \$2800?

Representing A's capital by x, the total capital is $3000 + x$, and A's profits are $2800 - x$. Substitute in the following:

Total capital : A's capital :: Total profits : A's profits.

Two Unknown Numbers

WRITTEN EXERCISES

Find the values of x and of y.

1. $x+2y=22$, $x^2+2y^2=162$.

SOLUTION

Given $x+2y=22$ (a), and $x^2+2y^2=162$ (b).

From (a), $x=22-2y$ (c).

Squaring (c), $x^2=484-88y+4y^2$ (d).

From (b), $x^2=162-2y^2$ (e).

Comparing (d) and (e),

$$484-88y+4y^2=162-2y^2 \quad (f).$$

Transposing, $6y^2-88y=-322$.

Dividing, $y^2-\frac{44y}{3}=-\frac{161}{3}$.

Completing the square,

$$y^2-\frac{44y}{3}+\frac{484}{9}=-\frac{483}{9}+\frac{484}{9}=\frac{1}{9}.$$

Extracting the square root,

$$y-\tfrac{22}{3}=\pm\tfrac{1}{3}.$$

$$\therefore y=\tfrac{23}{3}, \text{ or } \tfrac{21}{3}=7\tfrac{2}{3}, \text{ or } 7.$$

Substituting these values of y in (a).

When $y=7\frac{2}{3}$, $x+15\frac{1}{3}=22$, and $x=6\frac{2}{3}$ ⎫ *Ans.*
When $y=7$, $x+14=22$, and $x=8$ ⎭

Test: When $x=6\frac{2}{3}$ and $y=7\frac{2}{3}$, $x^2+2y^2=44\frac{4}{9}+117\frac{5}{9}=162$.

When $x=8$, and $y=7$, $x^2+2y^2=64+98=162$.

NOTE. As the values of x are obtained from (a), the values of x and y are tested in (b).

It will be observed that x and y have two values each, one set being $x=6\frac{2}{3}$, $y=7\frac{2}{3}$, and the other being $x=8$, $y=7$.

In this example the equation (f) contains but one unknown number, the other being eliminated by *comparison* of (d) and (e).

NOTE. The foregoing method, as well as the one given in solving the next example, is a general one applicable when one equation contains only the first power of each unknown number, and the other contains only their second powers.

2. $x^2 + 3y^2 = 31,\ 2x - y = 1.$

SOLUTION

Given $x^2 + 3y^2 = 31$ (a), and $2x - y = 1$ (b).

From (b), $y = 2x - 1$ (c).

Squaring (c), $y^2 = 4x^2 - 4x + 1$ (d).

Multiplying (d) by 3, $3y^2 = 12x^2 - 12x + 3$ (e).

Substituting in (a) this value of $3y^2$,

$$x^2 + 12x^2 - 12x + 3 = 31 \quad (f).$$

Simplifying, $13x^2 - 12x = 28$,

or, $$x^2 - \frac{12x}{13} = \frac{28}{13}.$$

Completing the square,

$$x^2 - \frac{12x}{13} + \frac{36}{169} = \frac{364}{169} + \frac{36}{169} = \frac{400}{169}.$$

Extracting the square root, $x - \frac{6}{13} = \pm\frac{20}{13}$

$$\therefore x = \tfrac{26}{13}, \text{ or } -\tfrac{14}{13} = 2, \text{ or } -1\tfrac{1}{13},$$

$$\text{and } y = 4 - 1, \text{ or } -2\tfrac{2}{13} - 1 = 3, \text{ or } -3\tfrac{2}{13}.$$

Ans. $x = 2,\ y = 3,$

$x = -1\tfrac{1}{13},\ y = -3\tfrac{2}{13}.$

Test both sets of values.

In equation (f), y is eliminated by substituting for $3y^2$ its value as found in (e). This method of elimination is called **elimination by substitution.**

3. $x + 2y = 5,$
   $2x^2 + y^2 = 6.$

4. $x^2 + 2y^2 = 24,$
   $2x + y = 10.$

5. $2x - y = 1,$
   $2x^2 + y^2 = 17.$

6. $4x^2 + 3y^2 = 183,$
   $2x + 3y = 27.$

7. $3x = 3y + 3,$
   $4x^2 = 4y^2 + 36.$

8. $2x + 5y = 29,$
   $2x^2 - 5y^2 = -117.$

9. $3x - 4y = 9,$
   $x^2 + y^2 = 58.$

10. $2x - y = 7,$
    $4x^2 - 2y^2 = 82.$

11. $x + 3y = 17,$
    $3x^2 - y^2 = 183.$

12. $4x + 5y = -14,$
    $x^2 - y^2 = -3.$

13. $2x^2 + 2y^2 = 104,\ 3x + 3y = 30.$

SOLUTION

Given $2x^2 + 2y^2 = 104$ (a), and $3x + 3y = 30$ (b).
Simplifying, (a) becomes $x^2 + y^2 = 52$ (c),
and (b) becomes $x + y = 10$ (d).
Squaring (d), $x^2 + 2xy + y^2 = 100$ (e).
Subtracting (c), $\underline{x^2 + y^2 = 52\ (c).}$
$2xy = 48$ (f).
Subtracting (f) from (c), $x^2 - 2xy + y^2 = 4.$
Extracting the square root, $x - y = 2$ (g),
and $x - y = -2$ (h).
Comparing (d) and (g), $2x = 12,\ 2y = 8.$
$\therefore x = 6,\ y = 4.$ *Ans.*
Comparing (d) and (h), $2x = 8,\ 2y = 12.$
$\therefore x = 4,\ y = 6.$ *Ans.*

Test: when $x = 6$, $y = 4$ the first member of (a) becomes
$2 \cdot 36 + 2 \cdot 16 \equiv 72 + 32 \equiv 104$;
when $x = 4$, $y = 6$ the first member of (a) becomes
$2 \cdot 16 + 2 \cdot 36 \equiv 32 + 72 \equiv 104$

14. $x - 2y = 1,\ x^2 + 4y^2 = 85.$

SOLUTION

Given $x - 2y = 1$ (a), and $x^2 + 4y^2 = 85$ (b)
Squaring (a), $x^2 - 4xy + 4y^2 = 1$ (c)
Subtracting (c) from (b), $4xy = 84$ (d)
Adding (b) and (d), $x^2 + 4xy + 4y^2 = 169$ (e)
Extracting the square root, $x + 2y = 13$ (f)
$x + 2y = -13$ (g)
Compare (a) and (f). Compare (a) and (g).

NOTE. This method is applicable when one equation is linear and the other is quadratic, and when the unknown numbers in the latter are the squares of the corresponding terms of the former after both equations are simplified.

15. $x^2+y^2=10,$
$x+y=4.$

16. $x^2+y^2=29,$
$x-y=-3.$

17. $x+y=-5,$
$x^2+y^2=13.$

18. $x+y=0,$
$x^2+y^2=32.$

19. $2x^2+2y^2=148,$
$3x-3y=-6.$

20. $x+y-1=0,$
$x^2+y^2-5=0.$

21. $4x^2+y^2-65=0,$
$2x-y-7=0.$

22. $x^2+5y^2=y^2+109,$
$x+3y=y+13.$

23. $x^2+10y^2=82+y^2,$
$3x+11y=30+2y.$

24. $4x^2+25y^2=200,$
$2x-5y=0.$

25. $9x^2+16y^2=337,\ 2xy+24=0.$

SOLUTION

Given $9x^2+16y^2=337$ (a), and $2xy+24=0$ (b).
Transposing (b), $2xy=-24$ (c).
Multiplying (c) by 12, $24xy=-288$ (d).
(a) $\underline{9x^2=16y^2=337}.$
Adding (a) and (d), $9x^2+24xy+16y^2=49$ (e).
Subtracting (d) from (a), $9x^2-24xy+16y^2=625$ (f).
Extracting square root of (e), $3x+4y=\pm 7$ (g).
Extracting square root of (f) $3x-4y=\pm 25$ (h).
that is,
$3x+4y=7\,(i), 3x-4y=25\,(j), 3x+4y=-7\,(k), 3x-4y=-25\,(l).$

Comparing (i) and (j), also (k) and (l), $x=\pm 5\frac{1}{3}$, $y=\mp 2\frac{1}{4}$.

Comparing (i) and (l), also (j) and (k), $x=\mp 3$, $y=\pm 4$, which means that when $x=5\frac{1}{3}$, $y=-2\frac{1}{4}$, and when $x=-5\frac{1}{3}$, $y=2\frac{1}{4}$; also that when $x=-3$, $y=4$, and when $x=3$, $y=-4$.

$$x=\pm 5\tfrac{1}{3} \text{ or } \pm 3,\ y=\mp 2\tfrac{1}{4} \text{ or } \mp 4.$$

Tests

When $x=5\frac{1}{3}$ and $y=-2\frac{1}{4}$, or $x=-5\frac{1}{3}$ and $y=2\frac{1}{4}$.
$9x^2+16y^2\equiv 9\cdot\frac{256}{9}+16\cdot\frac{81}{16}\equiv 256+81=337$
$2xy+24\equiv 2\cdot\frac{16}{3}(-\frac{9}{4})+24\equiv -24+24\equiv 0,$
or $2(-\frac{16}{3})(\frac{9}{4})+24\equiv -24+24\equiv 0.$

When $x=3$ and $y=-4$ or $x=-3$ and $y=4$.
$9x^2+16y^2\equiv 81+256\equiv 337,$
$2xy+24\equiv 2\cdot 3(-4)+24\equiv -24+24\equiv 0,$
or $2(-4)3+24\equiv -24+24\equiv 0.$

26. $x^2 + y^2 = 41$, $xy = 20$.

SOLUTION

Given $x^2 + y^2 = 41$ (a), and $xy = 20$ (b).
Adding (a) and twice (b), $x^2 + 2xy + y^2 = 81$ (c).
Subtracting twice (b) from (a), $x^2 - 2xy + y^2 = 1$ (d).
From (c), $x + y = \pm 9$ (e), From (d), $x - y = \pm 1$ (f).
that is, $x + y = 9$ (g), $x - y = 1$ (h), $x + y = -9$ (i), $x - y = -1$ (j).
Comparing (g) and (h), also (i) and (j), $x = \pm 5$, $y = \pm 4$.
Comparing (g) and (j), also (h) and (i), $x = \pm 4$, $y = \pm 5$.
$x = \pm 5$ or ± 4, $y = \pm 4$ or ± 5. *Ans.*

Equation (b) shows that x and y have the same signs, x being positive when y is positive, and *vice versa.*

In each of these two examples the terms containing the unknown numbers in one of the two simultaneous equations are exact squares, and the other equation has but one unknown term, which consists of the product of the unknown numbers.

27. $x^2 + y^2 = 113$,
$xy = 56$.

28. $x^2 + y^2 = 65$,
$3xy = 84$.

29. $x^2 + 4y^2 = 281$,
$4xy = 160$.

30. $4x^2 + y^2 = 52$,
$4xy - 48 = 0$.

31. $x^2 + 4y^2 = 40$,
$xy + 6 = 0$.

32. $4x^2 + 9y^2 = 244$,
$12xy = 240$.

33. $9x^2 + 4y^2 = 153$,
$3xy + 18 = 0$.

34. $4x^2 + 25y^2 = 641$,
$4xy = -40$.

35. $9x^2 + 25y^2 = 666$,
$5xy = -105$.

36. $16x^2 + 9y^2 = 1017$,
$6xy + 252 = 0$.

37. $4x^2 - 9y^2 = 28$, $2x - 3y = 14$.

SOLUTION

Given $4x^2 - 9y^2 = 28$ (a), and $2x - 3y = 14$ (b).
Dividing (a) by (b), member by member,
$2x + 3y = 2$ (c).
Comparing (b) and (c), $x = 4$, $y = -2$. *Ans.*

38. $x^2 - y^2 = 33,$
$x - y = 3.$

39. $x^2 - y^2 = 95,$
$x + y = 19.$

40. $x^2 - y^2 = 99,$
$x - y = 11.$

41. $x^2 - y^2 = 102,$
$x + y = 6.$

42. $x^2 - y^2 = 120,$
$x - y = 8.$

43. $x^2 - 4y^2 = 24,$
$x + 2y = 12.$

44. $4x^2 - 9y^2 = 36,$
$2x - 3y = 9.$

45. $9x^2 - 16y^2 = 85,$
$3x + 4y = 17.$

46. $4x^2 - 25y^2 = 99,$
$2x - 5y = 11.$

47. $9x^2 - 25y^2 = 78,$
$3x + 5y = 26.$

48. $x + y = 25,\ xy = 156.$

SOLUTION

Given $x + y = 25$ (*a*), and $xy = 156$ (*b*).
Squaring (*a*), $x^2 + 2xy + y^2 = 625$ (*c*).
Multiplying (*b*) by 4, $\underline{4xy = 624\ (d)}$.
Subtracting (*d*) from (*c*), $x^2 - 2xy + y^2 = 1.$
Extracting the square root, $x - y = \pm 1$ (*e*),
or, $x - y = 1$ (*f*), and $x - y = -1$ (*g*).
Comparing (*f*) and (*a*), $2x = 26$ and $2y = 24.$
$\therefore x = 13,\quad y = 12.$ *Ans.*
Comparing (*g*) and (*a*), $2x = 24$ and $2y = 26.$
$\therefore x = 12,\quad y = 13.$ *Ans.*

49. $2x - 3y = 33,\ xy = -42.$

SOLUTION

Given $2x - 3y = 33$ (*a*), and $xy = -42$ (*b*).
Squaring (*a*), $4x^2 - 12xy + 9y^2 = 1089$ (*c*).
Multiplying (*b*) by 24, $\underline{24xy = -1008\ (d)}$.
Adding (*c*) and (*d*), $4x^2 + 12xy + 9y^2 = 81.$
Extracting the square root, $2x + 3y = \pm 9$ (*e*),
or, $2x + 3y = 9$ (*f*), and $2x + 3y = -9$ (*g*).
Comparing (*f*) and (*a*), $4x = 42$ and $6y = -24.$
$\therefore x = 10\frac{1}{2},\quad y = -4.$ *Ans.*
Comparing (*g*) and (*a*), $4x = 24$ and $6y = -42.$
$\therefore x = 6,\quad y = -7.$ *Ans.*

50. $x + y = -2,$
$xy = -63.$

51. $x + 2y = -14,$
$2xy = 48.$

52. $2x + y = 17,$
$3xy = 90.$

53. $x + 3y = -23,$
$2xy = 28.$

54. $2x + 3y = 24,$
$xy = -30.$

55. $3x - 2y = 2,$
$4xy = 80.$

56. $2x + 5y = 41,$
$3xy = -54.$

57. $3x - 5y = 14,$
$2xy = -6.$

58. $4x + 3y = 7,$
$xy = -12.$

59. $2x - 7y = 17,$
$2xy = 12.$

WRITTEN PROBLEMS

1. The sum of the terms of a fraction is 7, and the sum of the squares of its terms is 25. What is the fraction?

2. Divide 169 into two such parts that the sum of their square roots may be 17.

Representing one part by x^2 and the other by y^2, the equations become $x^2 + y^2 = 169$ and $x + y = 17$.

3. The sum of the perimeters of two square fields is 96 rods, and their combined area is 290 square rods. Find the area of each.

$$4x + 4y = \text{sum of the perimeters.}$$

4. The product of two numbers is 60 and their difference is 7. What are the numbers?

$$xy = 60,\ x = 7 + y,\ \therefore (7 + y)y = 60.$$

5. Find two numbers whose sum is 10, the difference of their squares being 40.

6. The sum of the square roots of two numbers is 14 and the difference of their square roots is 2. Find the numbers.

Represent the numbers by x^2 and y^2.

7. A leaves M at the same time that B leaves N, and when they meet A has traveled 3 miles more than B. A reaches N 2 hours and B reaches M $4\frac{1}{2}$ hours after they meet. How far does each travel to the meeting place?

Let x = distance A goes *after*, and B goes before they meet.
$x+3$ = distance B goes *after*, and A goes before they meet.

Then $\frac{x}{2}$ = A's rate of traveling.
$\frac{x+3}{4\frac{1}{2}}$ = B's rate of traveling.

$(x+3) \div \frac{x}{2}$, the time required by A to reach meeting place,
is equal to $x \div \frac{x+3}{4\frac{1}{2}}$, the time required by B to reach meeting place.

8. The difference of the squares of the terms of a fraction divided by their difference is 10, and divided by their sum is 4. Find the fraction.

9. The sum of two numbers is 5, and the sum of their reciprocals is $\frac{5}{6}$. Find the numbers.

$x+y=5$ (*a*), $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$ (*b*). From (*a*) $x=5-y$. $\therefore \frac{1}{x}=\frac{1}{5-y}$ (*c*).
From (*b*) $\frac{1}{x}=\frac{5}{6}-\frac{1}{y}$, or $\frac{5y-6}{6y}$ (*d*). Compare (*c*) and (*d*).

10. The sum of two numbers is 10 and the sum of their reciprocals is $\frac{5}{8}$. What are the numbers?

11. The square of the sum of two numbers is 100 and the square of their difference is 4. Find the numbers.

12. A man has x dollars invested at 4 % and y dollars invested at 5 %, his annual interest from both sums being \$175. If he received 4 % on the latter sum and 5 % on the former, his annual income would be \$185. How many dollars has he at interest at each rate?

13. The square root of the sum of two numbers is 13. The difference of the numbers is 119. What are the numbers?